KNOPP'S CORE THEOREM AND SUBSEQUENCES OF A BOUNDED SEQUENCE

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Let s_n be a sequence in a p-dimensional Euclidean space E^p. Let $K_n = K(s_n, s_{n+1}, \cdots)$ be the convex hull of $s_n, s_{n+1}, \cdots$ and $\overline{K}_n$ its closure. The core of s_n is defined as $\bigcap_{n=1}^{\infty} \overline{K}_n$. Knopp's core theorem states that if $A = (a_{ij})$ is an infinite regular matrix with nonnegative elements, then the core of the A-transform of s_n is contained in the core of s_n. In particular if s_n is bounded, every A-limit of a subsequence of s_n is contained in the convex hull of limit points of s_n. With certain restrictions on A, the converse is also true; i.e., for any element ξ in the convex hull of limit points of s_n, there is a subsequence of s_n which is A-limitable to ξ. The main objective of this paper is to show that for any ξ in the convex hull of limit points of a bounded sequence s_n, there is a subsequence of s_n which is C_1- and E_1-limitable to ξ.

The following is Knopp's core theorem in E^p.

THEOREM 1. *Let s_n be a sequence in E^p and $A = (a_{ij})$ a regular matrix with $a_{ij} \geqq 0$. Let K_n be the convex hull of $s_n, s_{n+1}, \cdots$ and K_n' the convex hull of $s_n', s_{n+1}', \cdots$, where $s_n' = \sum_{j=1}^{\infty} a_{nj}s_j$ is defined for each $n = 1, 2, \cdots$. Then $\bigcap_{n=1}^{\infty} \overline{K}_n' \subset \bigcap_{n=1}^{\infty} \overline{K}_n$.*

PROOF. For each $\epsilon > 0$, define $K_{n\epsilon} = \{x \mid \inf_{y \in K_n} \|x - y\| \leqq \epsilon\}$. For given m and $\epsilon > 0$, choose $\nu(m, \epsilon)$ so that

$$\left\| \sum_{j=1}^{m-1} a_{nj}s_j \right\| < \epsilon, \quad \left| \sum_{j=1}^{\infty} a_{nj} - 1 \right| < \epsilon, \quad \left| \sum_{j=1}^{m-1} a_{nj} \right| < \epsilon \qquad \text{for } n \geqq \nu.$$

Now choose an index $k(n)$ so that

$$\left\| \sum_{j=m+k+1}^{\infty} a_{nj}s_j \right\| < \epsilon, \qquad \left| \sum_{j=m+k+1}^{\infty} a_{nj} \right| < \epsilon.$$

Then $\left| \sum_{j=m}^{m+k} a_{nj} - 1 \right| < 3\epsilon$. Assume $\epsilon < 1/3$ so that $\sum_{j=m}^{m+k} a_{nj} \neq 0$. Let $s_n' = \alpha_n + \beta_n + x_n\gamma_n$ where

$$\alpha_n = \sum_{j=1}^{m-1} a_{nj}s_j, \quad \beta_n = \sum_{j=m+k+1}^{\infty} a_{nj}s_j, \quad \gamma_n = \sum_{j=m}^{m+k} a_{nj}$$

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and

$$x_n = \frac{\sum_{j=m}^{m+k} a_{nj}s_j}{\sum_{j=m}^{m+k} a_{nj}}.$$

Since $x_n \in K_{m\epsilon}$ for every ϵ, $s_n'/\gamma_n \in K_{m\epsilon'}$ for $n \geqq \nu$ where $\epsilon' = 2\epsilon/(1-3\epsilon)$, $|\gamma_n - 1| < 3\epsilon$.

Now let $\xi \in \overline{K}_n'$ for every n. Then $\xi \in K'_{n\delta}$ for every $\delta > 0$ and every n; therefore $\xi = a_1 s'_{n_1} + \cdots + a_k s'_{n_k} + \eta$ where $\|\eta\| < \delta$, $n_i \geqq \nu$ for each i, $a_i > 0$ and $\sum_{i=1}^k a_i = 1$. Since $s_n'/\gamma_n \in K_{m\epsilon'}$ for $n \geqq \nu, \xi = \eta + \sum_{i=1}^k a_i x_{n_i}\gamma_{n_i}$ where $x_{n_i} \in K_{m\epsilon'}$ for $n_i \geqq \nu$ and $|\gamma_{n_i} - 1| < 3\epsilon$. Let $\gamma = \sum_{i=1}^k a_i\gamma_{n_i}$, then $(1/\gamma)(\sum_{i=1}^k a_i\gamma_{n_i}x_{n_i}) \in K_{m\epsilon'}$; hence $\xi \in \overline{K}_m$ for every m.

It is clear that $\bigcap_{n=1}^\infty \overline{K}_n$ always contains limit points of s_n; hence it contains the convex hull of limit points of s_n. Moreover if s_n is bounded $\bigcap_{n=1}^\infty \overline{K}_n$ is precisely the convex hull of limit points of s_n. To see this let Q be the set of limit points of s_n and $N_\epsilon(x)$ the ϵ-neighborhood of x. Then for each $\epsilon > 0$, $\bigcup_{x \in Q} N_\epsilon(x)$ contains all but a finite number of s_n. Now choose an index p so that $s_n \in \bigcup_{x\in Q} N_\epsilon(x)$ for $n \geqq p$. Moreover since $\xi \in K_{m\delta}$ for every $\delta > 0$ and m, $\xi = \sum_{i=1}^k a_i s_{n_i} + \eta$ where $a_i \geqq 0$, $\sum_{i=1}^k a_i = 1$, $n_i \geqq p$ and $\|\eta\| < \delta$. Therefore there are $\xi_1, \xi_2, \cdots, \xi_k$ in Q such that $\xi = \eta + \sum_{i=1}^k a_i\xi_i + \sum_{i=1}^k a_i\epsilon_i$ where $\|\epsilon_k\| \leqq \epsilon$; therefore, $\xi \in K_{\epsilon+\delta}(Q)$. Hence $\xi \in \overline{K}(Q)$. The above remark and Knopp's core theorem give the following corollary.

Corollary 1. *If A is an infinite regular matrix with nonnegative elements, then every A-limit of a bounded sequence is in the convex hull of the limit points of the sequence.*

The following is a sufficient condition on a regular matrix so that the converse of Corollary 1 holds.

Lemma 1. *Let A be a nonnegative regular matrix. Suppose there is a set of sequences $\mathbf{C}$ consisting of 0s and 1s with the following properties;*

(i) *For any $0 \leqq \alpha \leqq 1$, there is a sequence in $\mathbf{C}$ which is A-limitable to α.*

(ii) *If $\lim_{i\to\infty} \sum_{j=1}^\infty a_{ij}\epsilon_j = \lim_{i\to\infty} \sum_{k=1}^\infty a_{ij_k} = \alpha$ where $(\epsilon_i) \in \mathbf{C}$, then $\lim_{i\to\infty} \sum_{k=1}^\infty a_{ij_k}x_k = \alpha\beta$ if $(x_k) \in \mathbf{C}$ and is A-limitable to β.*

Then for any ξ in the convex hull of limit points of a bounded sequence s_n in E^p, there is a subsequence which is A-limitable to ξ.

Proof. Let Q be the set of limit points of s_n, and let $\xi = \sum_{i=1}^m a_i\xi_i$ where $\xi_i \in Q$, $a_i > 0$, and $\sum_{i=1}^m a_i = 1$. Proceed by induction on m. If

$m=1$, then simply choose a subsequence of s_n which converges to ξ. Let $a=\sum_{i=1}^{k-1} a_i$, $b_i=a_i/a$, $\eta=\sum_{i=1}^{k-1} b_i\xi_i$ so that $\xi=\eta a+a_k\xi_k$. There is a subsequence s_{n_ν} of s_n which is A-limitable to η. Also there is sequence ϵ_n in **C** so that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij}\epsilon_j = \lim_{i\to\infty} \sum_{k=1}^{\infty} a_{ij_k} = a.$$

Let

$$y_m = \begin{cases} s_{n_\nu} & \text{if } m = j_\nu, \\ \xi_k & \text{otherwise.} \end{cases}$$

Now let y'_m be the sequence obtained from y_m replacing the terms which equal ξ_k by successive elements of s_n which converges to ξ_k so that y'_m is a subsequence of s_n. By induction it is easy to see that y'_m thus constructed is A-limitable to ξ.

The next problem is to find a set **C** of Lemma 1 for C_1- and E_1-processes.

LEMMA 2. *There is a set of sequences* **C** *of Lemma* 1 *for* C_1- *and* E_1-*processes.*

PROOF. Let $\alpha\in[0, 1]$ and $\alpha=0\cdot a_1a_2\cdots$ be a binary representation of α. For each positive integer $k\geqq 2$ define a sequence $\overset{k}{s}_n$ as follows,

$$\overset{k}{s}_{2n-1} = 0, \quad \overset{k}{s}_{2(2n-1)} = 0, \cdots, \overset{k}{s}_{2^{k-1}(2n-1)} = 1, \quad \overset{k}{s}_{2^k n} = 0$$

and for $k=1$, define $\overset{1}{s}_{2n-1}=1$, $\overset{1}{s}_{2n}=0$. Then $\overset{k}{s}_n$ is C_1-limitable to $1/2^k$ for each k, and $\sum_{k=1}^{\infty} \overset{k}{s}_n=1$ for each n. Now the sequence defined by $s_n=\sum_{k=1}^{\infty} a_k\overset{k}{s}_n$ consists of the elements 0 and 1, and C_1-limitable to α. The condition (ii) of Lemma 1 can easily be verified. Note that every sequence used in the proof satisfies the condition $(s_1+\cdots+s_n)/n =\xi+o(1/\sqrt{n})$ where ξ is the C_1-limit of s_n; hence it is also E_1-limitable to ξ. Lemma 1 and Lemma 2 give the following theorem.

THEOREM 2. *Let* s_n *be any bounded sequence in* E^p. *Let* ξ *be any element in the convex hull of limit points of* s_n, *then there is a subsequence of* s_n *which is* C_1- *and* E_1-*limitable* (*hence also Abel and Borel limitable*) *to* ξ.

In general it is not possible to extend the result to unbounded sequences. For example $0, 1, 0, 2, 0, 3, \cdots$ has no C_1-limitable except the trivial ones. However, it can be shown that the sequence can be rearranged so that the resulting sequence is C_1-limitable to any pre-assigned nonnegative number. But the sequence 0, 1!, 0, 2!, 0, 3!, $\cdots$ does not even have this property. In general if s_n is a sequence having a limit point ξ and it has a subsequence $k_n \to \infty$ with $k_n/(k_1+\cdots+k_n)\to 0$, then for $\xi<\alpha<\infty$ it is possible to find a rearrangement of s_n whose C_1-limit is α. Without loss of generality assume $\xi=0$. Let $[(k_1+\cdots+k_n)/\alpha]$ be the greatest integer less than or equal to $(k_1+\cdots+k_n)/\alpha$. Construct a sequence y_n inserting 0s in $k_1, k_2, \cdots$ so that the number of 0s preceding k_n is $[(k_1+\cdots+k_n)/\alpha]$. Then y_n is C_1-limitable to α. Now there is a subsequence s_{n_k} of s_n which converges to 0. Replace each element in y_n which equals 0 by successive elements of s_{n_k}. Then the sequence y_n' thus constructed is C_1-limitable to α. Now insert the rest of the elements of s_n in y_n' occasionally so that the resulting sequence has C_1-limit α. This remark can be made also in E^p without much difficulty.

References

1. R. P. Agnew, *Cores of complex sequences and of their transforms*, Amer. J. Math. **61** (1939), 178–189.

2. H. G. Barone, *Limit points of sequences and their transforms by methods of summability*, Duke Math. J. **5** (1939), 740–752.

3. R. G. Cooke, *Infinite matrices and sequence spaces*, Macmillan, New York, 1950.

4. Konrad Knopp, *Zur Theorie der Limitierverfahren*. I, II, Math. Z. **31** (1930), 97–127, 276–305.

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