MULTIPLIER TRANSFORMATIONS. III

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Let $\boldsymbol{N}_\alpha$, $-1/2<\alpha<1/2$, be the space of those complex valued functions $F(n)$, $n\in\boldsymbol{I}$ the additive group of integers, for which $\boldsymbol{N}_\alpha[F]$ is finite where

$$\boldsymbol{N}_\alpha[F] = \left[\sum_{-\infty}^{\infty} |F(n)|^2(|n|+1)^{2\alpha}\right]^{1/2}.$$

If $F\in\boldsymbol{N}_0$ the Fourier transform

$$(M_2) \qquad F^\wedge(\theta) = \sum_{-\infty}^{\infty} F(n)e^{2\pi i n\theta}$$

is defined as a limit in the mean of order 2 for $\theta\in\boldsymbol{T}$, where $\boldsymbol{T}$ is the additive group of real numbers modulo 1. For such F the following inversion formula is valid,

$$F(n) = \int_{\boldsymbol{T}} F^\wedge(\theta)e^{-2\pi i n\theta}d\theta.$$

Let $t(\theta)$ be a bounded measurable function on $\boldsymbol{T}$ and let us define

$$TF\cdot(n) = \int_{\boldsymbol{T}} F^\wedge(\theta)t(\theta)e^{-2\pi i n\theta}d\theta$$

for $n\in\boldsymbol{I}$ and $F\in\boldsymbol{N}_0$. If

$$\boldsymbol{N}_\alpha[T] = \text{l.u.b.}\{\boldsymbol{N}_\alpha[TF]/\boldsymbol{N}_\alpha[F],\ F\in\boldsymbol{N}_\alpha\cap\boldsymbol{N}_0,\ F\not\equiv 0\}$$

is finite then, since $\boldsymbol{N}_\alpha\cap\boldsymbol{N}_0$ is dense in $\boldsymbol{N}_\alpha$, T has a unique extension as a bounded linear transformation (with norm $\boldsymbol{N}_\alpha[T]$) of $\boldsymbol{N}_\alpha$ into itself. The problem with which we are concerned is that of finding sufficient conditions on the multiplier function $t(\theta)$ which will insure that the corresponding multiplier transformation T is bounded on $\boldsymbol{N}_\alpha$. In the present paper, which continues investigations begun in [**1**; **2**; **3**], we will obtain a sufficient condition involving β-variation. A function $f(x)$ defined on $I=\{a\leqq x\leqq b\}$ is said to be of bounded β-variation ($1\leqq\beta<\infty$) if $\boldsymbol{V}_\beta[f,I]=\boldsymbol{V}_\beta[f]$ is finite where

$$\boldsymbol{V}_\beta[f] = \text{l.u.b.}\left[\sum_{k=0}^{n} |f(x_{k+1})-f(x_k)|^\beta\right]^{1/\beta}.$$

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Here the least upper bound is taken over all finite sets $a \leqq x_0 < x_1 < \cdots < x_n \leqq b$. Note that if $\beta_2 > \beta_1$

$$V_{\beta_2}[f]^{\beta_2} \leqq (2\|f\|_\infty)^{\beta_2-\beta_1} V_{\beta_1}[f]^{\beta_1}.$$

Thus if $V_{\beta_1}[f]$ is finite then so is $V_{\beta_2}[f]$. Our principal result is that if $V_\beta[t(\theta)]$ is finite, $\beta \geqq 2$, then $N_\alpha[T]$ is finite for $|\alpha| < 1/\beta$. This result is of interest in that the entire permissible range of α, $-1/2 < \alpha < 1/2$, is already obtained for $\beta = 2$.

We begin by recalling several results from [2] and [3]. Let $0 < \alpha < 1/2$ be fixed. Note that for $\alpha > 0$, $N_\alpha \subset N_0$ so that $F^\wedge(\theta)$ is well defined for every $F \in N_\alpha$. For $t(\theta)$ a bounded measurable function on T we define $A_\alpha[t]$ to be the smallest constant such that

$$\int_T \int_T |F^\wedge(\theta)|^2 |t(\theta) - t(\phi)|^2 |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\theta d\phi \leqq A_\alpha[t] N_\alpha[F]^2$$

for every $F \in N_\alpha$. $A_\alpha[t]$ may of course be $+\infty$.

LEMMA 1. *For* $0 < \alpha < 1/2$ *we have*

$$N_\alpha[T] \leqq 2 \operatorname{Max}\{\|t\|_\infty, A_\alpha[t]\}.$$

LEMMA 2. *For* $0 < \alpha < 1/2$ *there is a constant* $A(\alpha)$ *depending only on* α *such that for any* ψ *in* T

$$\int_T |F(\theta)|^2 |\sin \pi(\theta - \phi)|^{-2\alpha} d\theta \leqq A(\alpha) N_\alpha[F]^2.$$

LEMMA 3. *Let* $f(x)$ *be a real function defined on the interval* $I = \{a \leqq x \leqq b\}$. *For each* $\beta > 1$ *there exists a constant* $C(\beta)$ *depending only on* β *such that for each* f *for which* $V_\beta[f] < \infty$ *and each* $\epsilon > 0$ *there exists* $f_\epsilon(x)$ *with the properties*:

a. $\|f - f_\epsilon\|_\infty \leqq \epsilon \quad x \in I$;
b. $V_1[f_\epsilon] \leqq C(\beta) V_\beta[f]^\beta \epsilon^{1-\beta}$.

Here $\|\cdot\|_\infty$ is the uniform norm on I. This is proved in [2] under the added assumption that $f(x)$ is continuous. However a simple modification of the proof given there shows that this assumption is unnecessary.

LEMMA 4. *Suppose that* $0 < \alpha < 1/2$. *Let* $t(\theta)$ *be of bounded* 1-*variation on* T. *Then if* T *is the corresponding multiplier transformation we have*

$$N_\alpha[T]^2 \leqq B(\alpha)\{\|t\|_\infty^2 + \|t\|_\infty V_1[t]\},$$

where $B(\alpha)$ *is a finite constant depending only on* α.

PROOF. We begin by proving this result under the assumption that

$t(\theta)$ is in addition continuous. At the end this restriction will be removed, using a standard approximation argument. For $F \in \boldsymbol{N}_\alpha$ consider the quantity

$$Q = \int_T F^\wedge(\theta)^2 d\theta \int_T |t(\theta) - t(\phi)|^2 |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\phi$$

$$\leqq 2\|t\|_\infty \int_T |F^\wedge(\theta)|^2 d\theta \int_T |t(\theta) - t(\phi)| \, |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\phi.$$

We have

$$\int_T |t(\theta) - t(\phi)| \, |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\phi \leqq I_1 + I_2$$

where

$$I_1 = \int_\theta^{\theta+1/2} |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\phi \int_\theta^\phi |dt(\psi)| ,$$

$$I_2 = \int_{\theta-1/2}^{\theta} |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\phi \int_\phi^\theta |dt(\psi)| .$$

By Fubini's theorem

$$I_1 = \int_\theta^{\theta+1/2} |dt(\psi)| \int_\psi^{\theta+1/2} |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\phi.$$

An easy computation shows that there exists a constant $A_1(\alpha)$such that if $\theta \leqq \psi \leqq \theta + 1/2$

$$\int_\psi^{\theta+1/2} |\sin \pi(\theta - \phi)|^{-1-2\alpha} d\phi \leqq A_1(\alpha) |\sin \pi(\theta - \psi)|^{-2\alpha}.$$

Thus

$$I_1 \leqq A_1(\alpha) \int_\theta^{\theta+1/2} |\sin \pi(\theta - \psi)|^{-2\alpha} |dt(\psi)| ,$$

and similarly

$$I_2 \leqq A_1(\alpha) \int_{\theta-1/2}^{\theta} |\sin \pi(\theta - \psi)|^{-2\alpha} dt(\psi).$$

Making use of these inequalities and using Fubini's theorem we find that

$$Q \leqq 2A_1(\alpha)\|t\|_\infty \int_T |F^\wedge(\theta)|^2 d\theta \int_T |\sin \pi(\theta - \psi)|^{-2\alpha} |dt(\psi)|$$

$$\leqq 2\|t\|_\infty A_1(\alpha) \int_T |dt(\psi)| \int_T |F^\wedge(\theta)|^2 |\sin \pi(\theta - \psi)|^{-2\alpha} d\theta.$$

Applying Lemma 2 we obtain

$$Q \leqq 2A_1(\alpha)A(\alpha)\|t\|_\infty V_1[t]N_\alpha[F]^2.$$

Thus $A_\alpha[t] \leqq 2A_1(\alpha)A(\alpha)\|t\|_\infty V_1[t]$. Our proof is now complete if $t(\theta)$ is continuous. If $t(\theta)$ is not continuous we set

$$t_n(\theta) = \int_T k_n(\theta - \phi)t(\phi)d\phi \qquad n = 1, 2, \cdots$$

where $k_n(\theta)$ is any sequence of functions on T satisfying:

i. $k_n(\theta)$ is continuous;

ii. $k_n(\theta) \geqq 0$, $\int_T k_n(\theta)d\theta = 1$;

iii. $\lim_{n\to\infty} \int_U k_n(\theta)d\theta = 1$, for any fixed open set U in T which contains 0.

With these assumptions it is easily verified that:

i. $\|t_n\|_\infty \leqq \|t\|_\infty$;

ii. $V_1[t_n] \leqq V_1[t]$;

iii. $\lim_{n\to\infty} t_n(\theta) = t(\theta)$ for all θ at which $t(\cdot)$ is continuous.

Let T_n be the multiplier transform generated by $t_n(\theta)$; then

$$N_\alpha[T] \leqq \liminf_{n\to\infty} N_\alpha[T_n].$$

Since $t_n(\theta)$ is continuous

$$\begin{aligned} N_\alpha[T_n]^2 &\leqq B(\alpha)\{\|t_n\|_\infty^2 + \|t_n\|_\infty V_1[t_n]\}, \\ &\leqq B(\alpha)\{\|t\|_\infty^2 + \|t\|_\infty V_1[t]\}. \end{aligned}$$

Combining these results our desired lemma follows.

THEOREM. *Let $t(\theta)$ be defined for $\theta \in T$ and let T be the corresponding multiplier transformation. If $V_\beta[t]$ is finite (where $\beta > 2$) then*

$$N_\alpha[T] < \infty \qquad \text{if } |\alpha| < 1/\beta.$$

PROOF. By Lemma 3 there exists a sequence of functions $s_n(\theta)$, $\theta \in T$, such that

$$\begin{aligned} &\|s_n - t\|_\infty \leqq 2^{-n}, \\ &V_1[s_n] \leqq C(\beta)V_\beta[f]^\beta 2^{n(\beta-1)} \qquad n = 1, 2, \cdots. \end{aligned}$$

Let

$$\begin{aligned} t_1(\theta) &= s_1(\theta), \\ t_n(\theta) &= s_n(\theta) - s_{n-1}(\theta) \qquad n = 2, 3, \cdots. \end{aligned}$$

Then

$$t(\theta) = \sum_{1}^{\infty} t_n(\theta),$$

and thus by an evident argument

$$N_\alpha[T] \leqq \sum_{1}^{\infty} N_\alpha[T_n]$$

where T_n is the multiplier transformation generated by the multiplier function $t_n(\theta)$. We have

$$\|t_n\|_\infty = O(2^{-n}) \qquad n = 1, 2, \cdots,$$

$$V_1[t_n] = O(2^{n(\beta-1)}) \qquad n = 1, 2, \cdots.$$

Choose γ, $\alpha<\gamma<1/2$. By Lemma 4

$$N_\gamma[T_n] = O[(2^{-n})^2 + 2^{-n}2^{n(\beta-1)}]^{1/2}, \quad = O(2^{n(\beta/2-1)}).$$

On the other hand by Parseval's equality

$$N_0[T_n] = \|t_n\|_\infty = O(2^{-n}).$$

Applying the Riesz-Thorin convexity theorem we find that if $\alpha=(1-\theta)0+\theta\gamma$ then

$$N_\alpha[T_n] = O(2^{-n(1-\theta)}2^{n(\beta/2-1)\theta}), \quad = O(2^{n(-1+\beta\alpha/2\gamma)}).$$

Thus the series $\sum_1^\infty N_\alpha[T_n]$ is convergent if $\beta\alpha/2\gamma<1$; that is if $\alpha<2\gamma/\beta$. Since γ is arbitrary subject to the restriction $\alpha<\gamma<1/2$, it is always possible to choose γ so that $\alpha<2\gamma/\beta$ if $0<\alpha<1/\beta$. Thus our theorem is true if $0<\alpha<1/\beta$. The case $-1/\beta<\alpha<0$ follows by a familiar duality argument, while the case $\alpha=0$ is trivial.

For $f(x)$ defined on the interval I let $W_\beta[f, I]$ be the smallest constant such that for every $\epsilon>0$ there exists a function $f_\epsilon(x)$, $x\in I$, satisfying:

a. $\|f-f_\epsilon\|_\infty \leqq \epsilon$,

b. $V_1[f_\epsilon, I] \leqq W_\beta[f, I]\epsilon^{1-\beta}$.

$W_\beta[f, I]$ can of course be $+\infty$. Lemma 3 asserts that

$$W_\beta[f, I] \leqq C(\beta)V_\beta[f, I]^\beta. \tag{1}$$

The assumption in our principal theorem that $V_\beta[f, T]<\infty$ is made only to insure that $W_\beta[f, T]<\infty$. The following lemma shows that the assumption $W_\beta[f, T] < \infty$ is "almost" as strong as the assumption $V_\beta[f, T] < \infty$.

LEMMA 5. *For each β, $1 \leqq \beta < \infty$, and each $\gamma > \beta$ there exists a finite constant $A(\beta, \gamma)$ such that*

$$V_\gamma[f, I] \leqq A(\beta, \gamma)\|f\|_\infty^{(\gamma-\beta)/\gamma} W_\beta[f, I]^{1/\gamma}. \tag{2}$$

PROOF. For each $k = 0, 1, \cdots$ let f_k satisfy

$$\|f - f_k\|_\infty \leqq 2^{-k}\|f\|_\infty,$$
$$V_1[f_k] \leqq W_\beta[f] 2^{k(\beta-1)}\|f\|_\infty^{1-\beta}.$$

If we define

$$g_0(x) = f_0(x),$$
$$g_k(x) = f_k(x) - f_{k-1}(x) \qquad k = 1, 2, \cdots,$$

then

$$\sum_{k=0}^\infty g_k(x) = f(x) \qquad x \in I.$$

Moreover

$$\|g_k(x)\|_\infty \leqq 4\, 2^{-k}\|f\|_\infty,$$
$$V_1[g_k] \leqq 2W_\beta[f] 2^{k(\beta-1)}\|f\|_\infty^{1-\beta}.$$

It is easy to see using Hölder's inequality that $V_\gamma[f] \leqq \sum_0^\infty V_\gamma[g_k]$. Also

$$V_\gamma[g]^\gamma \leqq (2\|g\|_\infty)^{\gamma-1} V_1[g].$$

Thus

$$\begin{aligned} V_\gamma[f] &\leqq \sum_0^\infty (2^{3-k}\|f\|_\infty)^{(\gamma-1)/\gamma} (2W_\beta[f] 2^{k(\beta-1)}\|f\|_\infty^{1-\beta})^{1/\gamma} \\ &\leqq \|f\|_\infty^{(\gamma-\beta)/\gamma} W_\beta[f]^{1/\gamma} 2^{(3\gamma-2)/\gamma} \sum_0^\infty 2^{-k(\gamma-\beta)/\gamma} \\ &\leqq A(\beta, \gamma)\|f\|_\infty^{(\gamma-\beta)/\gamma} W_\beta[f]^{1/\gamma}. \end{aligned}$$

On the other hand the assumption $W_\beta[f] < \infty$ is slightly weaker than the assumption $V_\beta[f] < \infty$ in that for $\beta > 1$ no inequality of the form

$$V_\beta[f, I]^\beta \leqq A'(\beta) W_\beta[f, I] \tag{3}$$

is true for all f. To see this let us set $I_k=\{k-1\leqq x\leqq k\}$ for all $k=0, 1, 2, \cdots$, and $I^N=I_1\cup I_2\cup\cdots\cup I_N$. Further let

$$f_1(x) = 2^{-k/\beta}\sin[2^k(2\pi x)] \qquad x\in I_k.$$

Very simple computations show that there are positive constants $c_1(\beta)$ and $c_2(\beta)$ independent of k and N such that

$$W_\beta[f_1, I_k] \geqq c_1(\beta) \qquad k = 1, 2, \cdots,$$

$$W_\beta[f_1, I^N] \leqq c_2(\beta) \qquad N = 1, 2, \cdots.$$

It is evident that

$$\sum_1^N V_\beta[f, I_k]^\beta \leqq V_\beta[f, I^N]^\beta.$$

If (3) held then using (1) we would have

$$C(\beta)^{-1}\sum_1^N W_\beta[f, I_k] \leqq A'(\beta)W_\beta[f, I^N].$$

However for $f=f_1$ and for N sufficiently large this is impossible.

References

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