OPERATORS ON A QUASI-REFLEXIVE BANACH SPACE

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1. **Introduction.** Let X be a Banach space, $\mathfrak{B}(X)$ the closure of the algebra of all bounded linear operators on X of finite rank, the closure being taken in the topology of the norm

$$\|B\| = \sup_{x\in X} \frac{\|Bx\|}{\|x\|}, \qquad B \in \mathfrak{B}(X).$$

The present paper is concerned with a generalization of a theorem of F. Bonsall and A. W. Goldie [1] which states that if X is reflexive then $\mathfrak{B}(X)$ is an annihilator algebra.

It is shown that if X is quasi-reflexive the algebra $\mathfrak{B}(X)$ can be written as the direct sum of four closed subalgebras

$$\mathfrak{B}(X) = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3,$$

where $\mathfrak{A}_1$ is a right annihilator algebra which is annihilated on the right by the right ideal $\mathfrak{B}_2$ and $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ is a left annihilator algebra which is annihilated on the left by the nilpotent algebra $\mathfrak{B}_3$. Moreover if X is reflexive we have $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = (0)$, so that the above mentioned theorem is obtained as a special case of the present result.

2. **Definitions and notation.** Let X be a Banach space, X^* and X^{**} its first and second conjugate spaces. The symbol π will be used to denote the canonical isomorphism of X into X^{**}. The annihilator in X^* of a subspace Y of X will be denoted by Y^+.

If $x \in X$ and $x^* \in X^*$ then as in [4] we use the symbol $x \otimes x^*$ to denote the one dimensional operator on X defined by the equation

$$(x \otimes x^*)y = x^*(y)x \qquad \text{for each } y \in X.$$

Throughout this paper the closure of the algebra of bounded linear operators of finite rank will be denoted by $\mathfrak{B}(X)$. All algebras of operators under consideration are considered to be normed with the operator bound, that is

$$\|A\| = \sup_{x\in X} \frac{\|Ax\|}{\|x\|}\,.$$

If A is an operator on a Banach space X, we denote by A^* the

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adjoint of A, that is the operator on X^* defined by $A^*x^*(x)=x^*(Ax)$. The symbol I will denote the identity operator.

Let $\mathfrak{A}$ be an algebra and $\mathfrak{E}\subset\mathfrak{A}$. The right (left) annihilator of $\mathfrak{E}$ will be denoted by $R(\mathfrak{E})(L(\mathfrak{E}))$.

An algebra $\mathfrak{A}$ is called a *right* (*left*) *annihilator algebra* [4] if for every closed left (right) ideal $\mathfrak{J}$ we have $R(\mathfrak{J})=(0)$, $(L(\mathfrak{J})=(0))$ if and only if $\mathfrak{J}=\mathfrak{A}$.

A Banach space X is called *quasi-reflexive* of *order n* [2] if the quotient space $X^{**}/\pi X$ is (finite) n-dimensional.

3. Preliminary lemmas.

3.1. LEMMA. *Let X be a Banach space, Y_i, $i=1, 2$, closed linear subspaces of X such that $X=Y_1\oplus Y_2$. Let $\mathfrak{B}_i$ be the subset of $\mathfrak{B}(X)$ consisting of those operators in $\mathfrak{B}(X)$ with range contained in Y_i, $i=1, 2$. Then $\mathfrak{B}_i$ is a closed right ideal in $\mathfrak{B}(X)$ and $\mathfrak{B}(X)=\mathfrak{B}_1\oplus\mathfrak{B}_2$.*

PROOF. Let $A\in\mathfrak{B}_i$, $B\in\mathfrak{B}(X)$, then Range $AB\subset Y_i$, therefore $\mathfrak{B}_i$ is a right ideal. If $A_n\in\mathfrak{B}_i$ and $A_n\rightarrow A$ in the norm topology of $\mathfrak{B}(X)$, then for each $x\in X$, $A_nx\in Y_i$ and $A_nx\rightarrow A_x$ in the norm topology of X. Since Y_i is closed, $Ax\in Y_i$, so $A\in\mathfrak{B}_i$. Thus, it follows that $\mathfrak{B}_i$ is a closed right ideal in $\mathfrak{B}(X)$. Now, since $X=Y_1\oplus Y_2$ there exists a continuous projection P of X onto Y_1 with null space Y_2. If $B\in\mathfrak{B}(X)$ we can write $B=PB+(I-P)B$, where $PB\in\mathfrak{B}_1$ and $(I-P)B\in\mathfrak{B}_2$. This decomposition is obviously unique and therefore $\mathfrak{B}(X)=\mathfrak{B}_1\oplus\mathfrak{B}_2$.

LEMMA 2. *Let X, Y_1, $\mathfrak{B}_1$ be as in Lemma 1, let Z_i, $i=1, 2$ be closed linear subspaces of X^* such that $X^*=Z_1\oplus Z_2$. Let $\mathfrak{A}_i$ be the subset of $\mathfrak{B}_1$ whose elements are the operators in $\mathfrak{B}_1$ whose adjoints have range contained in Z_i, $i=1, 2$. Then $\mathfrak{A}_i$ is a closed left ideal in $\mathfrak{B}_1$ and $\mathfrak{B}_1=\mathfrak{A}_1\oplus\mathfrak{A}_2$.*

PROOF. The proof that $\mathfrak{A}_i$ is a closed left ideal is similar to the argument in Lemma 1 and hence is omitted. Let P be the continuous projection of X onto Y_1 with null space Y_2 and Q the continuous projection of X^* onto Z_1 with null space Z_2. Note first that $\mathfrak{A}_i$, $i=1, 2$, are closed subspaces of $\mathfrak{B}_1$ and that if $A_i\in\mathfrak{A}_i$, $i=1, 2$, we have $\|A_1\| =\|Q(A_1+A_2)\|\leq\|Q\|\,\|A_1+A_2\|$. It follows from Theorem 2.1 [3] that $\mathfrak{A}_1\oplus\mathfrak{A}_2$ is a closed subalgebra of $\mathfrak{B}_1$. Next, we show that $\mathfrak{A}_1\oplus\mathfrak{A}_2$ is dense in $\mathfrak{B}_1$. A one dimensional operator in $\mathfrak{B}_1$ is of the form $y\otimes x^*$ with $y\in Y$, $x^*\in X^*$ and can be written $y\otimes x^*=y\otimes Qx^*+y\otimes(I-Q)x^*$, with $y\otimes Qx^*\in\mathfrak{A}_1$ and $y\otimes(I-Q)x^*\in\mathfrak{A}_2$. It follows that every operator of finite rank in $\mathfrak{B}_1$ can be written as the sum of an operator in $\mathfrak{A}_1$ and an operator in $\mathfrak{A}_2$. Now, if $B\in\mathfrak{B}_1\subset\mathfrak{B}(X)$, there exists a sequence F_n of operators of finite rank on X such that $F_n\rightarrow B$ in the norm topology. Since P is a continuous projection on Y_1 and Range $B \subset Y_1$ we also have $PF_n \rightarrow B$ where PF_n is an operator of finite rank belonging to $\mathfrak{B}_1$ and therefore $PF_n \in \mathfrak{A}_1 \oplus \mathfrak{A}_2$. It follows that $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is dense in $\mathfrak{B}_1$ and therefore $\mathfrak{A}_1 \oplus \mathfrak{A}_2 = \mathfrak{B}_1$.

LEMMA 3. *If X is quasi-reflexive of order n then there exists Banach spaces X_1 and X_2 and an equivalent norm for X such that $X_2^* = X_1$, $X_1^* = X$ and*

$$X^* = X_1^{**} = \pi_1 X_1 \oplus (\pi_2 X_2)^+,$$
$$X = X_2^{**} = \pi_2 X_2 \oplus V^+,$$
$$X_1 = X_2^* = V \oplus U,$$

where π_i is the canonical embedding of X_i in X_i^{**}, $i = 1, 2$, and where U, V^+ and $(\pi_2 X_2)^+$ are all (finite) n-dimensional.

PROOF. Since X is quasi-reflexive of order n, it follows from Theorem 3.5 of [2] that X_1, X_2 exist such that $X_2^* = X_1$ and $X_1^* = X$ and such that both X_1 and X_2 are quasi-reflexive of order n. The direct sum decompositions follow at once from Theorem 3.3 and the proof of Theorem 3.1 of [2].

4. **Theorem.** *Let X be a quasi-reflexive space. Then $\mathfrak{B}(X) = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3$ where $\mathfrak{A}_1$ is a right annihilator algebra and $\mathfrak{B}_2$ a right ideal, which annihilates $\mathfrak{A}_1$ on the right; $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ is a left annihilator algebra and $\mathfrak{B}_3$ a nilpotent algebra which annihilates $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ on the left. Moreover the following are equivalent:*

(a) *$\mathfrak{A}_1$ is a left annihilator algebra,*

(b) *$\mathfrak{B}_i = (0)$, $i = 1, 2$, or 3,*

(c) *X is reflexive.*

PROOF. By Lemma 3 there exist Banach spaces X_1 and X_2 such that

$$X^* = \pi_1 V \oplus \pi_1 U \oplus (\pi_2 X_2)^+ \quad \text{and} \quad X = \pi_2 X_2 \oplus V^+.$$

Let $\mathfrak{A}_1$ denote the subalgebra of $\mathfrak{B}(X)$ whose elements are the operators in $\mathfrak{B}(X)$ with range in $\pi_2 X_2$ and which have adjoints with range in $\pi_1 V$; let $\mathfrak{B}_1$ denote the subalgebra of $\mathfrak{B}(X)$ whose elements are those operators in $\mathfrak{B}(X)$ with range in $\pi_2 X_2$, which have adjoints with range in $\pi_1 U$; let $\mathfrak{B}_2$ denote the right ideal of $\mathfrak{B}(X)$ whose elements are the operators in $\mathfrak{B}(X)$ with range in V^+; and let $\mathfrak{B}_3$ be the subalgebra of $\mathfrak{B}(X)$ consisting of those operators with range in $\pi_2 X_2$ and whose adjoints have range in $(\pi_2 X_2)^+$. Then an application of Lemma 1 and Lemma 2 yields

$$\mathfrak{B}(X) = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3.$$

Next we will show that $\mathfrak{A}_1$ is a right annihilator algebra. By the proof of Theorem 3.1 [2] we see that there exists in isomorphism α of π_2X_2 onto V^* such that $\alpha(\pi_2x_2)(v)=v(x_2)$ for all $x_2\in X_2$, $v\in V$.

We establish next an isomorphism β of $\mathfrak{A}_1$ onto $(\mathfrak{B}(V))^*$.[2] For $A\in\mathfrak{A}_1$ define $\beta A=\alpha A\alpha^{-1}$. If we let $T\in\mathfrak{B}(V)$ be defined by $Tv=\pi_1^{-1}A^*\pi_1v$ for all $v\in V$, then

$$(T^*v^*)(v) = v^*(\pi_1^{-1}A^*\pi_1v) = (A^*\pi_1v)(\alpha^{-1}v^*) = (\alpha A\alpha^{-1}v^*)(v),$$

and it follows that $\beta A=T^*$ and hence $\beta A\in(\mathfrak{B}(V))^*$.

The mapping β is onto since if $T^*\in(\mathfrak{B}(V))^*$ we can define $A\in\mathfrak{B}(X)$ by $A=\alpha^{-1}T^*\alpha P$, where P is the continuous projection of X onto π_2X_2, with null space V^+; clearly Range $A\subset\pi_2X_2$, and $\beta A=T^*\alpha P\alpha^{-1}=T^*$. To show that Range $A^*\subset\pi_1V$ we proceed as follows. Suppose first that T is a one-dimensional operator, then $T=v\otimes v^*$ for some $v\in V$, $v^*\in V^*$, and $T^*=v^*\otimes\pi v$, where π is the canonical embedding of V into V^{**}. We then have $A=\alpha^{-1}(v^*\otimes\pi v)\alpha P$ and for any $x^*\in X^*$, $x\in X$ we obtain $A^*x^*(x)=x^*[\alpha^{-1}(v^*\otimes\pi v)\alpha Px]=\pi v(\alpha Px)x^*(\alpha^{-1}v^*)=(Px)(v)x^*(\alpha^{-1}v^*)$ using the definition of α. Now $x=Px+(I-P)x$ and $(I-P)x\in V^+$, so $\pi_1v(x)=x(v)=Px(v)$, and hence $A^*x^*=x^*(\alpha^{-1}v^*)\pi_1v\in\pi_1V$. This shows that Range $A^*\subset\pi_1V$ in case T is a one-dimensional operator. Similarly if T is of finite rank we obtain Range $A^*\subset\pi_1V$. Finally, if T is arbitrary in $\mathfrak{B}(V)$, there exists a sequence T_n of operators of finite rank such that $T_n\to T$. Let $A_n=\alpha^{-1}T_n^*\alpha P$ and $A=\alpha^{-1}T^*\alpha P$. Since $T_n\to T$, since π_1V is a closed subspace of X^* and since Range $A_n^*\subset\pi_1V$ for each n, it follows that $A_n\to A$ and Range $A^*\subset\pi_1V$. The mapping β is therefore onto $(\mathfrak{B}(V))^*$.

It is clear furthermore that β is one to one, linear, bicontinuous and preserves multiplication, i.e., $\beta(AB)=(\beta A)(\beta B)$ for all A, $B\in\mathfrak{A}_1$. We can thus identify $\mathfrak{A}_1$ and $(\mathfrak{B}(V))^*$.

Now, $\mathfrak{B}(V)$ is a left annihilator algebra [4, p. 107] so $(\mathfrak{B}(V))^*$ is a right annihilator algebra. It follows that $\mathfrak{A}_1$ is a right annihilator algebra.

Next, if $A\in\mathfrak{A}_1$ and $B\in\mathfrak{B}_2$ then for any $x^*\in X^*$ and $x\in X$ we have $((AB)^*x^*)x=(A^*x^*)(Bx)=0$, since $A^*x^*\in\pi_1V$ and $Bx\in V^+$. Therefore $(AB)^*=0$ so that $AB=0$, hence $\mathfrak{B}_2$ annihilates $\mathfrak{A}_1$ on the right.

We consider next $\mathfrak{A}_1\oplus\mathfrak{B}_1$ which is the subalgebra consisting of those operators in $\mathfrak{B}(X)$ with range in π_2X_2 and whose adjoints have range in π_1X_1. In order to show that $\mathfrak{A}_1\oplus\mathfrak{B}_1$ is a left annihilator algebra it suffices to notice that $\mathfrak{A}_1\oplus\mathfrak{B}_1=(\mathfrak{B}(X_2))^{**}$. This equality follows from the fact that if $T\in\mathfrak{B}(X_2)$ and $T=x_2\otimes x_1$ with $x_2\in X_2$ and $x_1\in X_1=X_2^*$

[2] If $\mathfrak{B}$ is a set of operators we denote by $\mathfrak{B}^*$ the set $\{T^*|\,T\in\mathfrak{B}\}$.

then $T^{**}=\pi_2 x_2 \otimes \pi_1 x_1 \in \mathfrak{A}_1 \oplus \mathfrak{B}_1$, so that $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ and $(\mathfrak{B}(X_2))^{**}$ contain the same one-dimensional operators.

That $\mathfrak{B}_3$ annihilates $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ on the left follows from the fact that if $A \in \mathfrak{A}_1 \oplus \mathfrak{B}_1$, $B \in \mathfrak{B}_3$ $x^* \in X^*$ and $x \in X$ we have $((BA)^* x^*)(x) = (B^* x^*)(Ax) = 0$, since $B^* x^* \in (\pi_2 X_2)^+$ and $Ax \in \pi_2 X_2$, so that $(BA)^* = 0$ and therefore $BA = 0$. The same conclusion holds if $A \in \mathfrak{B}_3$, $\mathfrak{B}_3$ is therefore nilpotent.

Finally we notice that if $\mathfrak{A}_1$ is a left annihilator algebra it is an annihilator algebra and so are $(\mathfrak{B}(V))^*$ and $\mathfrak{B}(V)$, consequently by [1] V is a reflexive Banach space and so is V^*. But V^* is isomorphic with X_2 by proof of Theorem 3.1 [2]; so X_2 is reflexive and by Lemma 3 we conclude that X is reflexive. If X is reflexive then $U = V^+ = (\pi_2 X_2)^+ = (0)$ which implies $\mathfrak{B}_i = (0)$ for $i = 1$, 2 and 3.

If $\mathfrak{B}_i = (0)$ this implies either $U = (0)$ or $V^+ = (0)$ or $(\pi_2 X_2)^+ = (0)$. But by Lemma 3 either one of these inequalities implies the other two so that $\mathfrak{B}_1 = (0)$ and $\mathfrak{A}_1 \oplus \mathfrak{B}_1 = \mathfrak{A}_1$, and hence $\mathfrak{A}_1$ is a left annihilator algebra.

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