NOTE ON STAR-SHAPED SETS

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1. The aim of this note is to prove that if M is a compact subset of E^n and for some m every m-dimensional hyperplane through a fixed point $p \in E^n$ intersects M along a nonempty acyclic set, $1 \leqq m \leqq n-1$, then M is star-shaped with respect to p, i.e., if $a \in M$ then the segment $\overline{pa}$ is contained in M.

This theorem is a generalization of a theorem of Aumann [1]. We gave recently a proof of Aumann's theorem based on the theory of multivalent mappings (see [3]); the present proof follows essentially the same line. The topological lemma on which it is based is susceptible to further generalizations; however, we give it here only in its simplest and easily proved case which is needed for the proof of the theorem about star-shaped sets.

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2. $H_n(X)$ will denote the nth Čech homology group of the space X with the group Z_2 of integers mod 2 as the group of coefficients. We will say that X is acyclic if X is connected and $H_n(X)=0$, $n=1, 2, 3, \cdots$.

Let X be a compact metric space and let $\Phi: X \to 2^Y$ be an upper semi-continuous mapping of X into the space 2^Y of all nonempty compact subsets of a space Y. The triple $\mathfrak{F}=\{X, Y, \Phi\}$ will be called a family [2]. The set X will be called the basis of $\mathfrak{F}$, the sets $\Phi(x)$—the elements of $\mathfrak{F}$, the set $\bigcup_{x\in X} \Phi(x) \subset Y$—the field of $\mathfrak{F}$. The field will be also denoted by $\Phi(X)$. A family $\mathfrak{F}$ is said to be acyclic if all its elements are acyclic.

E^n will denote the Euclidean space, D^n the unit n-ball in E^n with center in the origin of coordinates o, S^{n-1} will denote the boundary of D^n. $E=E^k$, $2 \leqq k \leqq n-1$, will stand for a fixed k-dimensional Euclidean subspace of E^n, $E'=E^{n-k}$ will be the orthogonal complement of E in E^n.

For a fixed r, $1 \leqq r \leqq k-1$, $G_{k,r}$ will denote the grassmannian of (unoriented) r-planes in E^k. For every plane $x \subset E$ let $H(x)$ be the plane in E^n spanned by x and E'. If x runs through $G_{k,r}$ then the correspondence $x \to H(x)$ is a one-to-one correspondence between $G_{k,r}$ and the set of all $(n-k+r)$-planes in E^n containing E'.

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2.1. LEMMA. *Let* $\mathfrak{F}=\{G_{k,r}, E^n, \Phi\}$ *be an acyclic family satisfying* $(H(x)\cap S^{n-1})\subset\Phi(x)$ *for every* $x\in G_{k,r}$. *Then* $D^n\subset\Phi(G_{k,r})$.

PROOF. For $r=1$ the lemma was proved in [2, 4a]. (The sentence in parentheses on the bottom of p. 295 in [2] is incorrect; the proof however is correct, provided $m=2$.)

To prove it in the general case we consider a fixed $(n-k+r-1)$-plane E'' in E^n containing E'. Let

$$G' = \{x \in G_{k,r}: H(x) \supset E''\}.$$

Since the family $\mathfrak{F}$ restricted to G' is a family with $G_{k-r+1,1}$ as basis, we infer that $\Phi(G')\supset D^n$. Since $\Phi(G_{k,r})\supset\Phi(G')$, this proves the lemma.

2.2. LEMMA. *Let* $\mathfrak{F}=\{G_{k,r}, E^n, \Phi\}$ *be an acyclic family satisfying* $(E'\cap S^{n-1})\subset\Phi(x)\subset H(x)$ *for every* $x\in G_{k,r}$. *Then* $(E'\cap D^n)\subset\Phi(G_{k,r})$.

PROOF. We will consider the grassmannian $G_{k,k-r}$ of all $(k-r)$-planes in E. For every $x\in G_{k,k-r}$, x^* will denote the orthogonal complement of x in E, and $S(x)=x\cap S^{n-1}$. For any two sets A, $B\subset E^n$ let $A * B$ be the union of all segments $\overline{ab}$, $a\in A$, $b\in B$. It is obvious that if $B\subset H(x^*)$ then $S(x) * B$ is homeomorphic to the join of $S(x)$ with B. In particular, this implies

(i) If B is compact and acyclic and $B\subset H(x^*)$ then $S(x) * B$ is also compact and acyclic.

Let $h: E^n\to E^n$ be a homeomorphism of E^n onto itself satisfying the following conditions

(ii) $h(E'\cap D^n)\subset D^n$;

(iii) For every $x\in G_{k,k-r} h(S(x) * (E'\cap S^{n-1}))=H(x)\cap S^{n-1}$.

It is easy to construct such a homeomorphism.

Now, for every $x\in G_{k,k-r}$ we put $\Phi_1(x)=h(S(x) * \Phi(x^*))$. It follows from (i) that $\mathfrak{F}_1=\{G_{k,k-r}, E^n, \Phi_1\}$ is an acyclic family. Moreover, since $\Phi(x^*)\supset E'\cap S^{n-1}$ we have $S(x) * \Phi(x^*)\supset S(x) * (E'\cap S^{n-1})$ and (iii) implies $\Phi_1(x)\supset H(x)\cap S^{n-1}$. Therefore the family $\mathfrak{F}_1$ satisfies the conditions of Lemma 2.1 and we infer that

(iv) $D^n\subset\Phi_1(G_{k,k-r})$.

Let $y\in E'\cap D^n$. By (ii) and (iv) $h(y)\in\Phi_1(x)$ for some $x\in G_{k,k-r}$. Therefore $y \in h^{-1}(\Phi_1(x)) = S(x) * \Phi(x^*)$. Since $(S(x) * \Phi(x^*)) \cap E' =\Phi(x^*)\cap E'$ it follows that $y\in\Phi(x^*)$. Thus $(E'\cap D^n)\subset\Phi(G_{k,r})$ which completes the proof.

2.3. REMARK. Actually, a much stronger lemma holds. Namely, if $\mathfrak{F}=\{G_{k,r}, E^n, \Phi\}$ is an acyclic family satisfying $(E'\cap S^{n-1})\subset\Phi(x)$, then for some $x\in G_{k,r}$ $\Phi(x)\cap H^*(x)\neq 0$. This implies easily 2.2 and may be proved using methods from [3].

3. THEOREM. *Let $M \subset E^n$ be a compact set, and m a natural number, $1 \leqq m \leqq n-1$. If there exists a point $p \in E^n$ such that for every m-plane H through p, $H \cap M$ is acyclic then M is star-shaped with respect to p.*

PROOF. We remark first that it follows from [**3**, 2.1] that $p \in M$. Now let $a \in M$, $a \neq p$, and L be the line through a and p.

Suppose first that $m=1$. Then $a, p \in L \cap M$ and $L \cap M$ is connected. Thus $\overline{ap} \subset L \cap M$, which was to be proved.

Now let $2 \leqq m \leqq n-1$. Let S be the $(n-1)$-sphere in E^n such that $\overline{ap}$ is its diameter, let E be the $(n-1)$-plane in E^n orthogonal to L and passing through the midpoint of $\overline{ap}$.

For every $(m-1)$-plane x in E we define $\Phi(x) = H(x) \cap M$, where $H(x)$ is as before the m-plane in E^n spanned by x and L. Thus $H(x)$ passes through p and $\Phi(x)$ is acyclic. Therefore $\mathfrak{F} = \{G_{n-1,m-1}, E^n, \Phi\}$ is an acyclic family. Obviously, $S \cap L = \{a, p\} \subset \Phi(x) \subset H(x)$. Hence $\mathfrak{F}$ satisfies all conditions from Lemma 2.2 (with $k=n-1$, $r=m-1$) and it follows that $\overline{ap} \subset \bigcup \Phi(x) \subset M$. This completes the proof of the theorem.

REFERENCES

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