# NOTE ON STAR-SHAPED SETS

A. KOSINSKI[1]

1. The aim of this note is to prove that if $M$ is a compact subset of $E^n$ and for some $m$ every $m$-dimensional hyperplane through a fixed point $p \in E^n$ intersects $M$ along a nonempty acyclic set, $1 \leqq m \leqq n-1$, then $M$ is star-shaped with respect to $p$, i.e., if $a \in M$ then the segment $\overline{pa}$ is contained in $M$.

This theorem is a generalization of a theorem of Aumann [1]. We gave recently a proof of Aumann's theorem based on the theory of multivalent mappings (see [3]); the present proof follows essentially the same line. The topological lemma on which it is based is susceptible to further generalizations; however, we give it here only in its simplest and easily proved case which is needed for the proof of the theorem about star-shaped sets.

I have the pleasure to acknowledge my indebtedness to Dr. M. W. Hirsch for valuable suggestions.

2. $H_n(X)$ will denote the $n$th Čech homology group of the space $X$ with the group $Z_2$ of integers mod 2 as the group of coefficients. We will say that $X$ is acyclic if $X$ is connected and $H_n(X)=0$, $n=1, 2, 3, \cdots$.

Let $X$ be a compact metric space and let $\Phi: X \rightarrow 2^Y$ be an upper semi-continuous mapping of $X$ into the space $2^Y$ of all nonempty compact subsets of a space $Y$. The triple $\mathfrak{F}=\{X, Y, \Phi\}$ will be called a family [2]. The set $X$ will be called the basis of $\mathfrak{F}$, the sets $\Phi(x)$—the elements of $\mathfrak{F}$, the set $\bigcup_{x \in X} \Phi(x) \subset Y$—the field of $\mathfrak{F}$. The field will be also denoted by $\Phi(X)$. A family $\mathfrak{F}$ is said to be acyclic if all its elements are acyclic.

$E^n$ will denote the Euclidean space, $D^n$ the unit $n$-ball in $E^n$ with center in the origin of coordinates $o$, $S^{n-1}$ will denote the boundary of $D^n$. $E=E^k$, $2 \leqq k \leqq n-1$, will stand for a fixed $k$-dimensional Euclidean subspace of $E^n$, $E'=E^{n-k}$ will be the orthogonal complement of $E$ in $E^n$.

For a fixed $r$, $1 \leqq r \leqq k-1$, $G_{k,r}$ will denote the grassmannian of (unoriented) $r$-planes in $E^k$. For every plane $x \subset E$ let $H(x)$ be the plane in $E^n$ spanned by $x$ and $E'$. If $x$ runs through $G_{k,r}$ then the correspondence $x \rightarrow H(x)$ is a one-to-one correspondence between $G_{k,r}$ and the set of all $(n-k+r)$-planes in $E^n$ containing $E'$.

Received by the editors November 21, 1961.

[1] This work was supported by the National Science Foundation research grant NSF G-18919.

2.1. LEMMA. *Let* $\mathfrak{F}=\{G_{k,r}, E^n, \Phi\}$ *be an acyclic family satisfying* $(H(x)\cap S^{n-1})\subset\Phi(x)$ *for every* $x\in G_{k,r}$. *Then* $D^n\subset\Phi(G_{k,r})$.

PROOF. For $r=1$ the lemma was proved in [2, 4a]. (The sentence in parentheses on the bottom of p. 295 in [2] is incorrect; the proof however is correct, provided $m=2$.)

To prove it in the general case we consider a fixed $(n-k+r-1)$-plane $E''$ in $E^n$ containing $E'$. Let

$$G' = \{x \in G_{k,r}: H(x) \supset E''\}.$$

Since the family $\mathfrak{F}$ restricted to $G'$ is a family with $G_{k-r+1,1}$ as basis, we infer that $\Phi(G')\supset D^n$. Since $\Phi(G_{k,r})\supset\Phi(G')$, this proves the lemma.

2.2. LEMMA. *Let* $\mathfrak{F}=\{G_{k,r}, E^n, \Phi\}$ *be an acyclic family satisfying* $(E'\cap S^{n-1})\subset\Phi(x)\subset H(x)$ *for every* $x\in G_{k,r}$. *Then* $(E'\cap D^n)\subset\Phi(G_{k,r})$.

PROOF. We will consider the grassmannian $G_{k,k-r}$ of all $(k-r)$-planes in $E$. For every $x\in G_{k,k-r}$, $x^*$ will denote the orthogonal complement of $x$ in $E$, and $S(x)=x\cap S^{n-1}$. For any two sets $A$, $B\subset E^n$ let $A * B$ be the union of all segments $\overline{ab}$, $a\in A$, $b\in B$. It is obvious that if $B\subset H(x^*)$ then $S(x) * B$ is homeomorphic to the join of $S(x)$ with $B$. In particular, this implies

(i) If $B$ is compact and acyclic and $B\subset H(x^*)$ then $S(x) * B$ is also compact and acyclic.

Let $h\colon E^n\to E^n$ be a homeomorphism of $E^n$ onto itself satisfying the following conditions

(ii) $h(E'\cap D^n)\subset D^n$;

(iii) For every $x\in G_{k,k-r} h(S(x) * (E'\cap S^{n-1}))=H(x)\cap S^{n-1}$.

It is easy to construct such a homeomorphism.

Now, for every $x\in G_{k,k-r}$ we put $\Phi_1(x)=h(S(x) * \Phi(x^*))$. It follows from (i) that $\mathfrak{F}_1=\{G_{k,k-r}, E^n, \Phi_1\}$ is an acyclic family. Moreover, since $\Phi(x^*)\supset E'\cap S^{n-1}$ we have $S(x) * \Phi(x^*)\supset S(x) * (E'\cap S^{n-1})$ and (iii) implies $\Phi_1(x)\supset H(x)\cap S^{n-1}$. Therefore the family $\mathfrak{F}_1$ satisfies the conditions of Lemma 2.1 and we infer that

(iv) $D^n\subset\Phi_1(G_{k,k-r})$.

Let $y\in E'\cap D^n$. By (ii) and (iv) $h(y)\in\Phi_1(x)$ for some $x\in G_{k,k-r}$. Therefore $y \in h^{-1}(\Phi_1(x)) = S(x) * \Phi(x^*)$. Since $(S(x) * \Phi(x^*)) \cap E' =\Phi(x^*)\cap E'$ it follows that $y\in\Phi(x^*)$. Thus $(E'\cap D^n)\subset\Phi(G_{k,r})$ which completes the proof.

2.3. REMARK. Actually, a much stronger lemma holds. Namely, if $\mathfrak{F}=\{G_{k,r}, E^n, \Phi\}$ is an acyclic family satisfying $(E'\cap S^{n-1})\subset\Phi(x)$, then for some $x\in G_{k,r}$ $\Phi(x)\cap H^*(x)\neq 0$. This implies easily 2.2 and may be proved using methods from [3].

3. THEOREM. *Let* $M \subset E^n$ *be a compact set, and* $m$ *a natural number,* $1 \leqq m \leqq n-1$. *If there exists a point* $p \in E^n$ *such that for every* $m$*-plane* $H$ *through* $p$, $H \cap M$ *is acyclic then* $M$ *is star-shaped with respect to* $p$.

PROOF. We remark first that it follows from [3, 2.1] that $p \in M$. Now let $a \in M$, $a \neq p$, and $L$ be the line through $a$ and $p$.

Suppose first that $m=1$. Then $a$, $p \in L \cap M$ and $L \cap M$ is connected. Thus $\overline{ap} \subset L \cap M$, which was to be proved.

Now let $2 \leqq m \leqq n-1$. Let $S$ be the $(n-1)$-sphere in $E^n$ such that $\overline{ap}$ is its diameter, let $E$ be the $(n-1)$-plane in $E^n$ orthogonal to $L$ and passing through the midpoint of $\overline{ap}$.

For every $(m-1)$-plane $x$ in $E$ we define $\Phi(x) = H(x) \cap M$, where $H(x)$ is as before the $m$-plane in $E^n$ spanned by $x$ and $L$. Thus $H(x)$ passes through $p$ and $\Phi(x)$ is acyclic. Therefore $\mathfrak{F} = \{G_{n-1,m-1}, E^n, \Phi\}$ is an acyclic family. Obviously, $S \cap L = \{a, p\} \subset \Phi(x) \subset H(x)$. Hence $\mathfrak{F}$ satisfies all conditions from Lemma 2.2 (with $k=n-1$, $r=m-1$) and it follows that $\overline{ap} \subset \bigcup \Phi(x) \subset M$. This completes the proof of the theorem.

## REFERENCES


1. G. Aumann, *On a topological characterization of compact convex point sets*, Ann. of Math. **37** (1936), 443–447.

2. K. Borsuk and A. Kosinski, *Families of acyclic compacta in euclidean n-space*, Bull. Acad. Polon. Sci. Cl. III **3** (1955), 293–296.

3. A. Kosinski, *A theorem on families of acyclic sets and its applications*, Pacific J. Math. **12** (1962), 317–325.


UNIVERSITY OF CALIFORNIA, BERKELEY