REMARK ON THE POINCARÉ DUALITY THEOREM

WILLIAM BROWDER[1]

1. **Introduction.** In this note we study the homology properties of spaces X in which Poincaré duality holds, such as compact orientable manifolds. For such spaces we show that Poincaré duality holds also for the Bockstein spectral sequences (see [3]). As a consequence we get:

THEOREM 1. *Let X be a space with Poincaré duality*, $\dim X = 2m+1$, *m even. Then either* (1) $H_m(X) = F+T+T$ *or* (2) $H_m(X) = F+T+T +Z_2$, *where F is free abelian and T is a torsion group. In case* (2) $Sq^m: H^{m+1}(X; Z_2) \to H^{2m+1}(X; Z_2)$ *is not zero, so that the Stiefel-Whitney class $W \neq 0$. If $m = 2$ then $H_2(X) = F+T+T+Z_2$ if and only if $w_3 \neq 0$.*

THEOREM 2. *Let X be a space with Poincaré duality*, $\dim X = 2m+1$, *m even, $H^i(X; Z_2) = 0$ for $i < m$. Then $H_m(X) = F+T+T+Z_2$ if and only if $m = 2$ and $w_3 \neq 0$.*

Here $H_*(X)$ denotes the integral homology of X; Z_p denotes the integers mod p.

Similar results to these have been obtained by M. Rueff [4] using Seifert's "Linking invariant" [5].[2] C. T. C. Wall [7] has also studied a similar situation. Our methods have similarity to theirs, but use of the Bockstein spectral sequence instead of more conventional homology theories allows us to refine the results somewhat, bringing in Steenrod squares.

I am endebted to Wall for his comments, and to Emery Thomas who pointed out Lemma 7, enabling me to sharpen my original version of Theorem 2.

2. **Poincaré duality and the Bockstein spectral sequence.** All spaces X considered shall be assumed to have $H_i(X)$ finitely generated for all i.

DEFINITION. A space X is said to satisfy Poincaré duality if $H_n(X) = Z$, $H_i(X) = 0$ for $i > n$, and if $\mu \in H_n(X)$ is a generator, then $\cap \mu: H^q(X) \to H_{n-q}(X)$ is an isomorphism for all q. We will write $n = \dim X$.

We refer to [3, p. 48] for the definition of cap product. Let

$e \in C_n(X)$ be a cycle representing μ. Then $\cap e$ is a chain map (see [3; Lemma 7.5]) and $(\cap e)_* = \cap \mu$. (This requires a slight modification of the coboundary.)

We recall that the Bockstein spectral sequence of a chain complex C is the spectral sequence associated with the homology exact couple:

$$
\begin{array}{ccc}
H(C) & \xrightarrow{\quad i \quad} & H(C) \\
{}_{\partial_p}\nwarrow & & \swarrow_{j} \\
& H(C \otimes Z_p) &
\end{array}
$$

We refer to [3] for details of the definition and properties of the spectral sequence. In particular, a chain map induces a map of spectral sequences, and a chain map which induces homology isomorphism induces isomorphism of spectral sequences.

Hence, we may consider $\cap \mu$ as inducing isomorphism of the cohomology Bockstein spectral sequence in dimension q, $(E_{(r)}^q)$, with the homology Bockstein spectral sequence in dimension $n-q$, $(E_{n-q}^{(r)})$. By the usual argument, this is equivalent to the statement that the cup product pairing $E_{(r)}^q \otimes E_{(r)}^{n-q} \rightarrow E_{(r)}^n = Z_p$ is a nonsingular pairing.

LEMMA 1. $\dim(image\ d_r)^q = number\ of\ summands\ Z_{p^r}\ in\ H^q(C)$, $(d_r = the\ differential\ in\ the\ cohomology\ Bockstein\ spectral\ sequence\ E_{(r)})$.

This lemma follows immediately from the results of [3, §3].

LEMMA 2. If X satisfies Poincaré duality and $\dim X = n$, then $(image\ d_r)^n = 0$.

This follows immediately from Lemma 1, and the fact that $H^n(X) = Z$ is free.

LEMMA 3. Let X satisfy Poincaré duality, $\dim X = 2m+1$. If $x, y \in E_{(r)}^m$, then $(d_r x)y = (-1)^{m+1} x(d_r y)$.

PROOF. $xy \in E_{(r)}^{2m}$ so that $d_r(xy) = (d_r x)y + (-1)^m x(d_r y)$, which equals zero by Lemma 2, which proves Lemma 3.

Let $K_r = $ kernel of d_r in $E_{(r)}^m$, $I_r = $ image of d_r in $E_{(r)}^{m+1}$. Then I_r is orthogonal to K_r under the cup product pairing, from Lemma 3. Hence the cup product induces a nonsingular pairing of $E_{(r)}^m/K_r$ with I_r. Further d_r induces an isomorphism of $E_{(r)}^m/K_r$ with I_r so that we define the inner product A_r^p on I_r by $A_r^p(x, y) = (d_r^{-1}x)y$. (Where there is no ambiguity, we shall denote the inner product simply by $\langle , \rangle$.) Lemma 3 implies that if m is odd then $\langle x, y \rangle = \langle y, x \rangle$, i.e., $\langle , \rangle$ is symmetric.

LEMMA 4. *If m is even, and if either $p \neq 2$ or $p = 2$ and $r > 1$, then $\langle , \rangle$ is symplectic, i.e., $\langle x, x \rangle = 0$ for $x \in I_r$.*

PROOF. It follows from Lemma 3 that $\langle x, y \rangle = -\langle y, x \rangle$, which proves Lemma 4 if $p \neq 2$. If $p = 2$ and $r > 1$, we recall that by [3, Theorem 5.4], if $d_r x = y$, $x \in E_{(r)}^m$, m even, then $d_{r+1}\{x^2\} = \{xy\}$. Then by Lemma 2, $\{xy\} = 0$ in $E_{(r+1)}^n$, and since $E_{(r)}^n = E_{(r+1)}^n$, $xy = 0$ in $E_{(r)}^n$, and $\langle y, y \rangle = 0$.

LEMMA 5. *If m is even and $x \in H^m(X; Z_2)$, then $Sq^m Sq^1 x = x Sq^1 x = \langle Sq^1 x, Sq^1 x \rangle$.*

PROOF. By [3, Theorem 5.4], $d_2\{x^2\} = \{x Sq^1 x + Sq^m Sq^1 x\}$. Since $d_2\{x^2\} = 0$ by Lemma 2, $\{x Sq^1 x + Sq^m Sq^1 x\} = 0$ in $E_{(2)}$, hence $x Sq^1 x = Sq^m Sq^1 x$.

Next we define Stiefel-Whitney classes (S-W classes) w_k in $H^*(X; Z_2)$ after the method of Wu (see [8]). Namely, since Poincaré duality holds in $H^*(X; Z_2)$, the homomorphism $Sq^i: H^{n-i}(X; Z_2) \to H^n(X; Z_2)$ coincides with $u_i \cup$ for some $u_i \in H^i(X; Z_2)$. Set $U = \sum_i u_i$, $Sq = \sum_j Sq^j$. Define $W = \sum_k w_k$, $w_k \in H^k(X; Z_2)$ by $W = Sq U$. We get immediately:

LEMMA 6. *$W = 0$ if and only if $U = 0$ if and only if $Sq^i: H^{n-i}(X; Z_2) \to H^n(X; Z_2)$ is zero for all i.*

PROOF OF THEOREM 1. A vector space with a symplectic inner product is even dimensional (see [2]). Then it follows from Lemma 4 and Lemma 1 that if p is odd the number of summands Z_{p^r} in $H^{m+1}(X)$ (hence $H_m(X)$ by Poincaré duality) is even. This gives us alternatives (1) or (2). If $Sq^m: I_1 \to H^n(X; Z_2)$ is zero ($I_1 = (\text{image } Sq^1)^{m+1}$), then from Lemma 5, A_1^2 is symplectic and alternative (1) holds. Hence if alternative (2) holds, by Lemma 6, $W \neq 0$.

Since $(\text{image } Sq^1)^n = 0$ by Lemma 2, and $Sq^{2i+1} = Sq^1 Sq^{2i}$, it follows that only even squares reach dimension n and hence U is the sum of even dimensional elements. If $m = 2$, then $U = u_2$, for $Sq^4 = 0$ for dimensional reasons in $H^*(X; Z_2)$. If alternative (2) holds in this case, then $Sq^2: I_1 \to H^5(X; Z_2)$ is nonzero so that $U = u_2 \neq 0$ and $u_2 Sq^1 x = Sq^2 Sq^1 x \neq 0$, and $u_2 Sq^1 x = (Sq^1 u_2)x$ by Lemma 3. Hence $w_3 = Sq^1 u_2 \neq 0$.

Conversely if $m = 2$ and $w_3 \neq 0$, then $w_3 = Sq^1 u_2$. From Lemma 3 and the fact that A_1^2 is nonsingular it follows that $u_2 \cdot y \neq 0$ for some $y = Sq^1 x \in H^3(X; Z_2)$. Hence $Sq^2 y = u_2 \cdot y = Sq^2 Sq^1 x \neq 0$ and kernel $Sq^2: I_1 \to H^5(X; Z_2)$ is not all of I_1, so that dim (kernel Sq^2) = dim $I_1 - 1$ (since Sq^2 is a linear map into a space of dimension 1). Restricted to

kernel Sq^2, A_1^2 is symplectic by Lemma 5, so that dim I_1 is odd and alternative (2) holds. This completes the proof of Theorem 1.

Theorem 2 will follow from Theorem 1 and the following lemma:

LEMMA 7. *If* $w_i = 0$ *for* $i < m$, *m even* > 2, *then* $w_{m+1} = 0$.

PROOF. By a result of J. F. Adams [1] relations holding among S-W classes in differentiable manifolds, hold also for our case. If $m = 2l$, according to the formula of Wu [9],[3] $Sq^l w_{l+1} = w_{m+1} + D$, where D is a product of w_i's for $i \leq m$. Since $w_i = 0$ for $i < m$, $D = 0$ so that $Sq^l w_{l+1} = w_{m+1}$. If $l > 1$, then $l + 1 < 2l$ so that $w_{l+1} = 0$ and hence $w_{m+1} = 0$.

PROOF OF THEOREM 2. First let us suppose $H_m(X) = F + T + T + Z_2$. Then A_1^2 is not symplectic and $Sq^m Sq^1 x \neq 0$ for some $x \in H^m(X; Z_2)$. As in the proof of the last part of Theorem 1, $U = u_m$ and $u_m Sq^1 x = Sq^m Sq^1 x \neq 0$. But $u_m Sq^1 x = (Sq^1 u_m)x$, so that $Sq^1 u_m \neq 0$ and $W = u_m + Sq^1 u_m$ so that $w_m = u_m$ and $w_{m+1} = Sq^1 u_m \neq 0$. Hence $m = 2$ by Lemma 7.

The converse follows immediately from Theorem 1.

Example of differentiable manifolds which satisfy the hypotheses of Theorem 2 are described in [6].

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[3] This fact does not require the full strength of Wu's formula, but can be seen very easily by considering the cohomology suspension $\sigma^*: H^*(B_0; Z_2) \to H^*(0; Z_2)$. It is well known that $\sigma^*(w_i) = h_{i-1}$ where $\{h_m\}$ are a simple system of generators for $H^*(0; Z_2)$, the w_i are a set of generators for $H^*(B_0; Z_2)$ and kernel $\sigma^* = $ decomposable elements. Then the result we need follows from the relation $Sq^k h_k = h^2_k = h_{2k}$ in $H^*(0; Z_2)$.