

Thus every neighbourhood of x contains points y such that $y \in B$ and $P(y, A) = 0$. Hence

$$0 \leq P(y, A_1) \leq P(y, A) = 0.$$

But this contradicts the continuity of $P(x, A)$.

BIBLIOGRAPHY

1. K. L. Chung, *Markov chains*, Springer, Berlin, 1960.
2. N. Dunford and J. Schwartz, *Linear operators*, Interscience, New York, 1958.
3. P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.
4. S. Kakutani and K. Yosida, *Operator theoretical treatment of Markov processes and mean ergodic theorem*, Ann. of Math. (2), **42** (1941), 188–228.
5. J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.

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DARBOUX FUNCTIONS OF BAIRE CLASS ONE AND DERIVATIVES

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Introduction. Let $I_0 = [0, 1]$ and let R be the reals. Let B_1 be the class of functions $f: I_0 \rightarrow R$ of Baire type at most one, and denote by D the class of functions $f: I_0 \rightarrow R$ which possess the Darboux property, i.e., take connected sets into connected sets. The class $B_1 \cap D$ is abbreviated by (B_1, D) . If Δ is the class of functions $f: I_0 \rightarrow R$ which are derivatives, then we have the well-known relation $\Delta \subset (B_1, D)$. It is of interest to have characterizations for the classes Δ and (B_1, D) . In this paper two characterizations of (B_1, D) are given as well as a characterization of Δ . This characterization together with one characterization of (B_1, D) provides a measurement by how much a function in (B_1, D) may fail to be in Δ .

Throughout the paper we will use the following notation. For $A \subset I_0$, A° is the interior of A relative to I_0 , \bar{A} stands for the closure of A , and $|A|$ denotes the Lebesgue measure of A .

First characterization of (B_1, D) . We have occasion to use the following characterizations of B_1 . (1) $f \in B_1$ if and only if for each $a \in R$ the sets $\{x: f(x) \geq a\}$, $\{x: f(x) \leq a\}$ are G_δ ; (2) $f \in B_1$ if and only if every perfect subset P of I_0 has a point of continuity of $f|P$ (f restricted to P) [3].

Presented to the Society January 25, 1962; received by the editors January 8, 1962.

¹ Supported by National Science Foundation Grant G-18920.

THEOREM 1. $f \in (B_1, D)$ if and only if for each $a \in R$, the sets $\{x: f(x) \geq a\}$, $\{x: f(x) \leq a\}$ are G_δ with compact components.

PROOF. Let $f \in (B_1, D)$, and let $a \in R$. By (1) we only need to show that the components of $\{x: f(x) \geq a\}$, $\{x: f(x) \leq a\}$ are compact. Let Q be a component of $E = \{x: f(x) \geq a\}$. We may suppose that Q is a nondegenerate interval with endpoints $\alpha < \beta$. If $f(\alpha) < a$, then the set $f([\alpha, \beta])$ would be disconnected, contradicting $f \in D$. Therefore, $\alpha \in E$ and similarly $\beta \in E$. Consequently, $Q = [\alpha, \beta]$, and the proof of the necessity is complete.

For the sufficiency we observe first that $f \in B_1$ by (1). If we deny $f \in D$, we have an interval $[\alpha, \beta] \subset I_0$ such that $f([\alpha, \beta])$ is not connected. For notational simplification we may assume that $[\alpha, \beta] = I_0$. There exists $a \in R$ such that the sets $E_1 = \{x: f(x) \leq a\}$, $E_2 = \{x: f(x) \geq a\}$ satisfy $I_0 = E_1 \cup E_2$, $E_1 \neq \emptyset$, $E_2 \neq \emptyset$, and $E_1 \cap E_2 = \emptyset$. Let $\{Q\}_1$ be the collection of components of E_1 which are nondegenerate, and let $\{Q\}_2$ be defined analogously. Let $\{Q\} = \{Q\}_1 \cup \{Q\}_2$. By hypothesis, each $Q \in \{Q\}$ is compact, and two distinct components in $\{Q\}$ are disjoint. Hence the set $P = I_0 - \bigcup Q^0$ is perfect, where the union is extended over all $Q \in \{Q\}$. By (2) there is a point $x_0 \in P$ at which $f|_P$ is continuous. Since $I_0 = E_1 \cup E_2$, we may assume that $x_0 \in E_1$, and since $E_1 \cap E_2 = \emptyset$, we infer that $f(x_0) < a$. Therefore, there is a $\delta > 0$ such that $I_\delta \cap P \subset E_1$, where $I_\delta = [x_0 - \delta, x_0 + \delta] \cap I_0$.

Let $Q \in \{Q\}_2$. We will prove that $I_\delta \cap Q = \emptyset$. For, if $I_\delta \cap Q \neq \emptyset$, there is an endpoint γ of Q which is in I_δ . Since $\gamma \in E_2 \cap P$, this contradicts $I_\delta \cap P \subset E_1$. Since every $x \in E_2 - \bigcup Q^0$ is in P , where the union is extended over all $Q \in \{Q\}_2$, we infer that $I_\delta \cap E_2 = \emptyset$, and hence $I_\delta \subset E_1$. There is then $Q \in \{Q\}_1$ such that $I_\delta \subset Q$, and thus $x_0 \in Q^0$, contradicting $x_0 \in P$. This completes the proof.

REMARK. In the necessity of Theorem 1 we have actually proved that for every $f \in D$ the sets $\{x: f(x) \geq a\}$, $\{x: f(x) \leq a\}$ have compact components. Without the condition that these sets be G_δ the converse is not true as the example $f(x) = 1$, x rational, $f(x) = 0$, x irrational, shows.

APPLICATIONS. Theorem 1 can be applied to show that various derivatives are in (B_1, D) . As an application, we will consider approximate derivatives and n th Peano derivatives. For the properties of approximate derivatives that will be used see [2], and for those of the n th Peano derivative see [4]. In these papers it has been shown that approximate derivatives and n th Peano derivatives are in (B_1, D) . We will show that Theorem 1 provides a very simple proof of this fact.

THEOREM 2. *If $f: I_0 \rightarrow R$ is an approximate derivative or n th Peano derivative, then $f \in (B_1, D)$.*

PROOF. It is known that $f \in B_1$ [2; 4]. If Q is a component of $\{x: f(x) \geq a\}$, simple arguments used in the above mentioned papers show that $\bar{Q} \subset \{x: f(x) \geq a\}$, and thus the components of $\{x: f(x) \geq a\}$ are compact.

Theorem 1 can be applied to provide a simple proof of a characterization of (B_1, D) due to Zahorski [5]. Let $f: I_0 \rightarrow R$, and let for $a \in R$, $E_a = \{x: f(x) > a\}$, $B_a = \{x: f(x) < a\}$.

DEFINITION (ZAHORSKI). $f \in M_0 (f \in M_1)$ if and only if for each $a \in R$ the sets E_a, B_a are F_σ and each point of E_a, B_a is a bilateral point of accumulation (condensation) of E_a, B_a , respectively.

THEOREM 3. $M_0 = M_1 = (B_1, D)$.

PROOF. The inclusion $M_1 \subset M_0$ is obvious, and $(B_1, D) \subset M_1$ is easy to prove. It suffices to show that $M_0 \subset (B_1, D)$. Since $f \in M_0$ implies that $f \in B_1$, for each $a \in R$, the sets $\{x: f(x) \geq a\}$, $\{x: f(x) \leq a\}$ are G_δ . Let Q be a component of $E = \{x: f(x) \geq a\}$. We may assume that Q is an interval with endpoints $\alpha < \beta$. If $f(\alpha) < a$, the point α would not be a right point of accumulation of $\{x: f(x) < a\}$. Hence $f(\alpha) \geq a$, and similarly, $f(\beta) \geq a$. Thus $Q = \bar{Q}$, and by Theorem 1 the proof is complete.

Second characterization of (B_1, D) . Let $\{I\}$ be the collection of all nondegenerate compact subintervals of I_0 . We shall use the notation $I \rightarrow x$ to denote $x \in I$ and $|I| \rightarrow 0$. Following Gleyzal [1] we say that an interval function $\phi: \{I\} \rightarrow R$ converges on I_0 if and only if for each $x \in I_0$, $\lim_{I \rightarrow x} \phi(I)$ exists. In [1], Gleyzal proved the following characterization of $B_1: f \in B_1$ if and only if f is the limit of a convergent interval function.

DEFINITION. A function $f: I_0 \rightarrow R$ is said to possess *property (C_1)* if and only if for each $I \in \{I\}$ there exists $x_I \in I^0$ such that $I \rightarrow x$ implies that $f(x_I) \rightarrow f(x)$, for each $x \in I_0$.

REMARK. A comparison with ordinary continuity may clarify the condition (C_1) . A function $f: I_0 \rightarrow R$ is continuous on I_0 if and only if $(*) I \rightarrow x, x_I \in I^0$ implies $f(x_I) \rightarrow f(x)$, for each $x \in I$. Condition (C_1) is a formulation of the *weakest* version of $(*)$ in the sense that we only require that in each I^0 there is a point x_I so that $I \rightarrow x$ implies $f(x_I) \rightarrow f(x)$.

THEOREM 4. $f \in (B_1, D)$ if and only if f has *property (C_1)* .

PROOF. First, suppose that f has property (C_1) . If $\phi(I) = f(x_I)$, then $\phi(I) \rightarrow f(x)$ as $I \rightarrow x$. Consequently, by Gleyzal's theorem, $f \in B_1$. In order to show that $f \in D$, we only need to show, by Theorem 1, that the sets $\{x: f(x) \geq a\}$, $\{x: f(x) \leq a\}$ have compact components. Let Q be a component of $E = \{x: f(x) \geq a\}$. We may assume that Q is a nondegenerate interval with endpoints $\alpha < \beta$. Let $x_n \in (\alpha, \beta)$ with $x_n \rightarrow \alpha$, and let $I_n = [\alpha, x_n]$. Then $I_n \rightarrow \alpha$, and thus $f(x_{I_n}) \rightarrow f(\alpha)$. Since $x_{I_n} \in I_n^0 \subset Q$, we have $f(x_{I_n}) \geq a$, and thus $f(\alpha) \geq a$. Similarly, $f(\beta) \geq a$, and hence $Q = \bar{Q}$.

The necessity is more complicated. We assume that $f \in (B_1, D)$. Since $f \in B_1$, there is a sequence of continuous functions $f_n: I_0 \rightarrow \mathbb{R}$ such that $f_n \rightarrow f$ on I_0 . For each n , select a finite number of intervals I_{n1}, \dots, I_{nj_n} in $\{I\}$ such that $I_0 = I_{n1}^0 \cup \dots \cup I_{nj_n}^0$, $|I_{nk}| < 1/n$, $k = 1, \dots, j_n$, and the saltus of f_n on each I_{nk} is $< 1/n$. Define $\{I\}_1 = \{I_{nk}, k = 1, \dots, j_n; n = 1, 2, \dots\}$. We note that $\{I\}_1$ covers I_0 in the sense of Vitali.

Let $\{I\}_2$ be the collection of all $I \in \{I\}$ for which there exists $J \in \{I\}_1$ such that $I \subset J$, and let $\{I\}_3$ be the collection of the remaining intervals. For $I \in \{I\}_2$, let n_0 be the largest integer for which there exists k , $1 \leq k \leq j_{n_0}$, such that $I \subset I_{n_0k}$. Among the k 's for which $I \subset I_{n_0k}$ select one k_0 , and call (n_0, k_0) the pair associated with I .

Let $I \in \{I\}_2$ and let (n, k) be the pair associated with I . Since $f \in D$, $\overline{f(I)}$ is a closed interval, and since f_n is continuous, $f_n(I_{nk})$ is a compact interval. Hence there are points $y_{nk} \in I_{nk}$, $\xi_I \in \overline{f(I)}$ such that

$$|f_n(y_{nk}) - \xi_I| = \text{dist}[f_n(I_{nk}), f(I)].$$

Since $f \in D$, we can choose $x_I \in I^0$ such that $|\xi_I - f(x_I)| < 1/n$. If $I \in \{I\}_3$, select any $x_I \in I^0$. For each $I \in \{I\}$ we have now chosen a point $x_I \in I^0$. We will verify that $f(x_I) \rightarrow f(x)$ if $I \rightarrow x$. We will proceed in several steps.

STEP 1. Let $I \in \{I\}_2$ and let (n, k) be the pair associated with I . Then $I \subset I_{nk}$. We define $\phi_1(I) = f_n(y_{nk})$. We will show that ϕ_1 converges to f on I_0 . This part of the proof is essentially a reproduction of Gleyzal's argument [1] with some modifications. Let $\xi \in I_0$ and let $\{I_j\} \subset \{I\}_2$ be a sequence with $I_j \rightarrow \xi$. For each n , there is a $j_n > 0$ such that for $j > j_n$, $I_j \subset I_{nk}$ for some k . Hence, for $j > j_n$, $|I_{nj_k j}| < 1/n$, where (n_j, k_j) is the pair associated with I_j . Since $n_j \geq n$, we have that $n_j \rightarrow \infty$ and $I_{n_j k_j} \rightarrow \xi$ as $j \rightarrow \infty$.

Let p be a positive integer. There is $l > 0$ such that $|f_n(\xi) - f(\xi)| < 1/p$ for $n > l$. There is $l' > 0$ such that $j > l'$ implies that $n_j > \max(p, l)$.

Let $j > \max(l', l)$. Then

$$\begin{aligned} |\phi_1(I_j) - f(\xi)| &= |f_{n_j}(y_{n_j k_j}) - f(\xi)| \\ &\leq |f_{n_j}(y_{n_j k_j}) - f_{n_j}(\xi)| + |f_{n_j}(\xi) - f(\xi)| \\ &< \frac{1}{n_j} + \frac{1}{p} < \frac{2}{p}. \end{aligned}$$

Hence $\phi_1(I_j) \rightarrow f(\xi)$.

STEP 2. For $I \in \{I\}$, let $\phi(I) = f(x_I)$, where $x_I \in I^0$ is the point chosen above. We will prove that ϕ converges to f on I_0 . Let $\xi \in I_0$ and let $\{I_j\} \subset \{I\}$ be a sequence with $I_j \rightarrow \xi$. There is $j_1 > 0$ such that $j > j_1$ implies $I_j \subset I_{1k}$ for some k . Hence for $j > j_1$, $I_j \in \{I\}_2$. Consequently, by step 1, $\phi_1(I_j) \rightarrow f(\xi)$. For $j > j_1$, let (n_j, k_j) be the pair associated with I_j . Then $I_j \subset I_{n_j k_j}$, $I_{n_j k_j} \rightarrow \xi$, and $\phi_1(I_j) = f_{n_j}(y_{n_j k_j})$. Thus

$$\begin{aligned} |\phi(I_j) - \phi_1(I_j)| &= |f(x_{I_j}) - f_{n_j}(y_{n_j k_j})| \\ &\leq |f(x_{I_j}) - \xi_{I_j}| + |\xi_{I_j} - f_{n_j}(y_{n_j k_j})|, \end{aligned}$$

where ξ_{I_j} has been defined previously. Thus

$$|\phi(I_j) - \phi_1(I_j)| < \frac{1}{n_j} + \text{dist}[f(I_j), f_{n_j}(I_{n_j k_j})] \leq \frac{1}{n_j} + |f_{n_j}(\xi) - f(\xi)|.$$

Since $f_{n_j}(\xi) \rightarrow f(\xi)$ as $j \rightarrow \infty$, we infer that $|\phi(I_j) - \phi_1(I_j)| \rightarrow 0$. Hence $\phi(I_j) \rightarrow f(\xi)$, completing the proof.

Characterization of Δ . As before, let $\{I\}$ be the collection of all nondegenerate compact intervals of I_0 .

DEFINITION. A function $f: I_0 \rightarrow R$ is said to have property (C_2) if and only if for each $I \in \{I\}$ there is $x_I \in I^0$ such that

- (1) $I \rightarrow x$ implies that $f(x_I) \rightarrow f(x)$, for each $x \in I_0$,
- (2) if $I = I_1 \cup I_2$, $I_1^0 \cap I_2^0 = \emptyset$, then

$$f(x_I) = \frac{f(x_{I_1}) |I_1| + f(x_{I_2}) |I_2|}{|I_1| + |I_2|}.$$

REMARK. Condition (1) is the same as (C_1) . Condition (2) says that $f(x_I)$ is between $f(x_{I_1})$ and $f(x_{I_2})$. One may refer to (2) by saying that $f(x_I)$ is *mean-valued*. We note that (2) is the same as saying that $f(x_I) |I|$ is an additive interval function.

THEOREM 5. $f \in \Delta$ if and only if f has property (C_2) .

PROOF. Suppose $f \in \Delta$. Then there exists $F: I_0 \rightarrow R$ such that on I_0 , $F'(x) = f(x)$. Let $I = [a, b]$ be in $\{I\}$. Then there exists $x_I \in I^0$

such that $F(b) - F(a) = f(x_I)|I|$. Thus the interval function $f(x_I)|I|$ is additive. If $I_j \rightarrow x$, $I_j = [a_j, b_j]$, then

$$f(x_{I_j}) = \frac{F(b_j) - F(a_j)}{b_j - a_j} \rightarrow f(x).$$

Conversely, assume that f has property (C_2) . Then, for each $I \in \{I\}$ there is $x_I \in I^0$ with the properties (1) and (2). Define $F(0) = 0$ and for $0 < x \leq 1$, define $F(x) = f(x_I)|I|$, where $I = [0, x]$. Since $f(x_I)|I|$ is an additive interval function, it follows that for $I = [a, b]$,

$$F(b) - F(a) = f(x_I)(b - a).$$

Let $x \in I_0$ and let $\{I_j\} \subset \{I\}$ be a sequence with $I_j \rightarrow x$, and set $I_j = [a_j, b_j]$. Then, since $f(x_{I_j}) \rightarrow f(x)$, we infer that

$$\frac{F(b_j) - F(a_j)}{b_j - a_j} \rightarrow f(x).$$

Thus $F'(x) = f(x)$, and the proof is complete.

REMARK. Comparing Theorem 5 with Theorem 4, we observe that the condition (2) in (C_2) is precisely the property by which a Darboux function of Baire class one may fail to be a derivative.

It is possible to rewrite Theorem 5 in a slightly different form. An interval function $\phi: \{I\} \rightarrow R$ will be termed *mean-valued* if and only if $\phi(I)|I|$ is additive.

THEOREM 6. $f \in \Delta$ if and only if f is the limit of a convergent and mean-valued interval function.

The proof is the same as the one above.

BIBLIOGRAPHY

1. A. Gleyzal, *Interval functions*, Duke Math. J. **8** (1941), 223-230.
2. C. Goffman and C. J. Neugebauer, *On approximate derivatives*, Proc. Amer. Math. Soc. **11** (1960), 962-966.
3. I. P. Natanson, *Theorie der Funktionen einer reellen Veränderlichen*, Akademie-Verlag, Berlin, 1954.
4. H. W. Oliver, *The exact Peano derivative*, Trans. Amer. Math. Soc. **76** (1954), 444-456.
5. Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1-54.

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