A REMARK ON COMMUTABLE FUNCTIONS AND CONTINUOUS ITERATIONS

MAREK KUCZMA

The purpose of this note is to apply some recent results of B. Choczewski to a question concerning commutable functions and continuous iterations. Namely, we shall prove the following:

THEOREM. *Let $f(x)$ be a function of class C^2 in an interval $(a, b\rangle$, such that $f(x)>x$ in (a, b), $f(b)=b$, $\lim_{x\to a+0} f(x)=a$, $f'(x)>0$ in $(a, b\rangle$, and $f'(b)<1$.*[1] *Moreover let us assume that the function $(d^2/dx^2)f^{-1}(x)$ satisfies a Lipschitz condition in a left neighbourhood of the point b ($f^{-1}(x)$ denotes the inverse function of $f(x)$, which exists since $f'(x)>0$).*

Then there exists a unique one-parameter family of functions $\phi_s(x)$, $s\in(-\infty, \infty)$, satisfying the following conditions for all s:

(A) *$\phi_s(x)$ is of class C^2 in $(a, b\rangle$.*

(B) *$\phi_s(x)\in(a, b\rangle$ for $x\in(a, b\rangle$, $\phi_s(b)=b$.*

(C) *$\phi_s(x)$ commutes with $f(x)$, i.e.*

$$f[\phi_s(x)] = \phi_s[f(x)]. \tag{1}$$

The family of the functions $\phi_s(x)$ is also, in a sense, the "best" family of continuous iterations of the function $f(x)$. The functions $\phi_s(x)$ may be obtained as the limit of the sequence defined by

$$\phi_{s,n}(x) = f^{-n}(\phi_{s,0}[f^n(x)]),\ x \in (a, b\rangle, \qquad n = 1, 2, 3, \cdots, \tag{2}$$

where $f^k(x)$ denotes the kth iterate of the function $f(x)$, i.e., $f^0(x)=x$, $f^{k+1}(x)=f[f^k(x)]$, $f^{k-1}(x)=f^{-1}[f^k(x)]$, $k=0, \pm 1, \pm 2, \cdots$, and $\phi_{s,0}(x)$ is an arbitrary function of class C^2 in $(a, b\rangle$ such that

$$\phi_{s,0}(x) \in (a, b\rangle \text{ for } x \in (a, b\rangle,\ \phi_{s,0}(b) = b,\ \phi'_{s,0}(b) = [f'(b)]^s. \tag{3}$$

Furthermore, if the function $f(x)$ is of class $C^r (r>2)$ in $(a, b\rangle$, then the functions $\phi_s(x)$ are also of class C^r in $(a, b\rangle$.

PROOF. In order to simplify the notation we write

$$h(x) \overset{df}{=} f^{-1}(x).$$

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[1] The number a may also equal $-\infty$; but b must be finite, which undoubtedly is a defect of the present theorem. The condition $\lim_{x\to a+0} f(x)=a$ may be omitted, but then the functions $\phi_s(x)$ occurring in the thesis of the theorem may not be defined in the whole interval $(a, b\rangle$, but only in a left neighbourhood of the point b. The conditions $f(x)>x$ in (a, b), $f'(b)<1$, may be replaced by $f(x)<x$ in (a, b), $f'(b)>1$. And naturally, the endpoints, a and b, of the interval in question may be interchanged.

Then equation (1) takes the form

$$(4)\qquad \phi(x) = h(\phi[f(x)]).$$

Moreover, we have

$$(5)\qquad h'(b)[f'(b)]^2 = f'(b) < 1.$$

Consequently, as a direct consequence of a theorem of B. Choczewski [2], it follows that for every number $s \in (-\infty, \infty)$ there exists exactly one function $\phi_s(x)$ satisfying equation (4), and fulfilling conditions (A), (B), and the condition[2]

$$(6)\qquad \phi_s'(b) = [f'(b)]^s.$$

(Condition (C) is fulfilled, since the functions $\phi_s(x)$ satisfy equation (4), which is equivalent to (1)).

It follows from the uniqueness of ϕ_s that $\phi_k(x) = f^k(x)$ for integral k. In particular we have $\phi_1(x) = f(x)$.

The paper by B. Choczewski has not yet been published; and since the method employed by him is essential to our further considerations, we shall present here a very short outline of his arguments, naturally omitting all details.[3]

Let $\mathfrak{R}$ be the space of all functions $\phi(x)$ of class C^2 in the interval $\langle b-\eta, b\rangle$ (where η is a sufficiently small, fixed positive number, depending only on $f(x)$ and s), fulfilling the conditions:

$$(7)\qquad \begin{gathered} \phi(x) \in \langle a+\alpha, b\rangle \text{ for } x \in \langle b-\eta, b\rangle, \quad \phi(b) = b, \quad \phi'(b) = [f'(b)]^s \\ |\phi''(x)| \leqq K \qquad \text{for } x \in \langle b-\eta, b\rangle, \end{gathered}$$

(where α and K are suitably chosen positive numbers, depending only on $f(x)$ and s), and endowed with the metric

$$\rho[\phi, \psi] = \sup_{\langle b-\eta,\, b\rangle} |\phi''(x) - \psi''(x)|,$$

(with which $\mathfrak{R}$ becomes a complete metric space). Consider the transformation

$$\Phi[\phi] = h(\phi[f(x)]).$$

This transformation (under a suitable choice of η, α and K) maps $\mathfrak{R}$

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[2] Of course, for $s \in (-\infty, \infty)$ the values of $[f'(b)]^s$ run over the set of all positive numbers. However, in order that $\phi_s(x)$ form a family of continuous iterations of $f(x)$ (cf. (9) below), it is convenient to define $\phi_s(x)$ with the aid of relation (6).

[3] In his paper [2] B. Choczewski deals with the more general equation $\phi(x) = H(x, \phi[f(x)])$.

into itself. Furthermore, in view of (5), we have, for $\phi, \psi \in \mathcal{R}$

$$\rho[\Phi[\phi], \Phi[\psi]] < \vartheta\rho[\phi, \psi], \qquad \vartheta < 1.$$

Consequently, Φ is a contraction mapping, and by the theorem of Banach-Caccioppoli, has exactly one fixed point in $\mathcal{R}$, i.e. there exists exactly one function $\phi_s(x)$ of class C^2 in $\langle b-\eta, b\rangle$, satisfying equation (4) and conditions (7). This function can be uniquely extended onto the whole interval $(a, b\rangle$ in such a manner that it will satisfy equation (4).

Hence it follows (see also [**3**, Theorem V]) that the function $\phi_s(x)$ can be obtained as the limit of the sequence defined by the recurrence formula

$$\phi_{s,n+1}(x) = h(\phi_{s,n}[f(x)]), \qquad n = 0, 1, 2, \cdots, \tag{8}$$

where $\phi_{s,0}(x)$ is an arbitrary function which, when restricted to $\langle b-\eta, b\rangle$, belongs to $\mathcal{R}$, and thus e.g. an arbitrary function of class C^2 in $(a, b\rangle$, fulfilling conditions (3). One can easily verify that formula (2) follows immediately from (8).

From the theorems of B. Choczewski [**2**] it further follows that the function $\phi_s(x)$ has the same regularity properties as $f(x)$.

To complete the proof of our theorem it remains only to show that the functions $\phi_s(x)$ form a family of continuous iterations of $f(x)$, i.e. that they satisfy the relation

$$\phi_s[\phi_t(x)] = \phi_{s+t}(x), \quad x \in (a, b\rangle, \quad s, t \in (-\infty, \infty). \tag{9}$$

Now, the function $\phi_s[\phi_t(x)]$, just like $\phi_s(x)$ and $\phi_t(x)$, is of class C^2 in $(a, b\rangle$ and evidently commutes with $f(x)$. We have also $\phi_s[\phi_t(x)] \in (a, b\rangle$ for $x \in (a, b\rangle$, $\phi_s[\phi_t(b)] = b$, $(d/dx)\phi_s[\phi_t(x)]\big|_{x=b} = \phi_s'[\phi_t(b)]\phi_t'(b) = \phi_s'(b)\phi_t'(b) = [f'(b)]^{s+t}$. But the function with these properties is unique. Thus $\phi_s[\phi_t(x)] = \phi_{s+t}(x)$, which was to be proved.

REMARK. The above theorem is closely related to some results of L. Berg [**1**]. The results obtained by Berg are, in some respect, stronger than ours—the condition $b < \infty$ does not occur, and the case $f'(b) = 1$ is also discussed. On the other hand, in most of the theorems in [**1**] assumptions of analyticity, or at least of multiple differentiability of $f(x)$, as well as the assumption that $f''(x)$ has a constant sign, occur; while in our case the assumptions regarding the regularity of $f(x)$ are weaker.

Similar equations have been also investigated by S. Sternberg [**5**]. He has obtained similar results, however in a different way.

It follows from [**4**] that in condition (A) it is not sufficient to require only the continuity of $\phi_s(x)$: for equation (1) admits continuous solution depending on an arbitrary function. Unfortunately, however, our method does not allow us to decide whether in addition to $\phi_s(x)$ there also exist other functions of class C^1 in $(a, b\rangle$ and fulfilling conditions (B) and (C). (For the linear function; $f(x)=b+q(x=b)$ such functions do not exist; cf. [1, §1].)

References

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Uniwersytet Jagielloński, Kraków, Poland