BOUNDEDNESS OF LIMITS

CHIN-SHUI HSÜ

This is an answer to a problem of A. Wilansky [2]. Suppose \( \sum b_k \) converges absolutely. Out of each infinite sequence \( x = \{ x_n \} \) we may construct

\[
\beta_1(x) = x_1, \quad \beta_n(x) = x_n + x_{n-1} + \sum_{k=1}^{n-1} b_k x_k \quad (n > 1).
\]

Obviously \( ||x|| = \text{l.u.b.} \{ |\beta_1(x)|, |\beta_2(x)|, \cdots \} \) forms a norm over the linear space of all convergent sequences. The problem is whether the linear functional \( \lim x = \lim_{n \to \infty} x_n \) is bounded in this norm.

We may solve (1) to have

\[
x_n = \sum_{r=1}^{n} (-1)^{n-r} g^{(r)}_n \beta_r(x)
\]

where

\[
g^{(n)}_n = 1, \quad g^{(r)}_n = \begin{vmatrix}
1 + b_r & 1 & 0 & \cdots & 0 \\
b_r & 1 + b_{r+1} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_r & b_{r+1} & b_{r+2} & \cdots & 1 + b_{n-1}
\end{vmatrix} \quad (r < n).
\]

On expanding \( g^{(r)}_n \) along the diagonal we have \( g^{(r)}_n = 1 + \text{linear combination of products of } b_r, \cdots, b_{n-1} \) with factors of \( (1 - b_r)(1 - b_{r+1}) \cdots (1 - b_{n-1}) \) as coefficients. Hence

\[
|g^{(r)}_n| < B, \quad |g^{(r)}_n - 1| < C(|b_r| + \cdots + |b_{n-1}|)
\]

where \( B, C \) are independent of \( n \) and \( r \).

By (3) we also have

\[
g^{(r+1)}_n = b_r g^{(r)}_n = \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 \\
1 & 1 + b_{r+2} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & b_{r+2} & b_{r+3} & \cdots & 1 + b_{n-1}
\end{vmatrix}
\]

Received by the editors October 13, 1961.

1 Editorial note: After this paper had been accepted for publication a specific counterexample to Wilansky’s conjecture was published by Hayman and Wilansky [1].
\[ g^{(r)}_n - g^{(r)}_{n-1} = b^{(r)}_{n-1} g_{n-1} - b^{(r)}_{n-2} g_{n-2} + \cdots + (-1)^{n-r} b^{(r)}_{r+1} g_{r+1} + (-1)^{n-r-1} b_r. \]

Expanding (5) along the diagonal and arguing as in (4),

\[ \left| g^{(r)}_n - g^{(r+1)}_n \right| < D \left| b_r \right| \]

where \( D \) is independent of \( n \) and \( r \).

Let

\[ z_1 = x_1, \quad z_n = x_n + x_{n-1}, \quad \beta_{n,r} = \beta_r - \beta_{r+1} + \cdots + (-1)^{n-r} \beta_n, \]

then

\[ z_n = k_n^{(1)} \beta_1(x) + \cdots + k_n^{(n-1)} \beta_{n-1}(x) + \beta_n(x) \]

where

\[ k_n^{(r)} = (-1)^{n-r} g^{(r)}_{n-1} + (-1)^{n-r-1} g^{(r)}_{n-2} = -b_r + b_{r+1} g_{r+1} + \cdots \]

\[ + (-1)^{n-r} b_{n-1} g_{n-1}. \]

Since \( \beta_{r+2, r+1} = x_{r+1} - x_{r+2} + (b_r x_r + \cdots + b_{r+2} x_{r+2}) \), we have

\[ |\beta_{n,r}| < E \]

where \( E \) is independent of \( n \) and \( r \). Then

\[ \left| \sum_{r=1}^{n-1} k_n^{(r)} \beta_r(x) \right| = \left| k_n^{(r+1)} \beta_{r+1} + \sum_{r=1}^{n-2} (k_n^{(r)} + k_n^{(r+1)}) \beta_{n,r} + k_n^{(n-1)} \beta_{n,n-1} \right| \]

\[ \leq E \{ (2C + 2D) \left( \left| \beta_{r+1} \right| + \cdots + \left| \beta_{n-1} \right| \right) + 2C \left| b_{n-1} \right| \}. \]

Hence, from (8),

\[ \lim x = \frac{1}{2} \left[ \lim_{n \to \infty} \beta_n(x) + \sum_{r=1}^{\infty} k^{(r)}(r) \beta_r(x) \right] \]

where

\[ k^{(r)} = \lim_{n \to \infty} k_n^{(r)} = \sum_{n=0}^{\infty} (-1)^{r-1} g_{r+1} b_{r+1}. \]

**Theorem.** The linear functional \( \lim \) is bounded if and only if \( \sum_{r=1}^{\infty} |k^{(r)}| < \infty \).

**Proof.** If \( \sum_{r=1}^{\infty} |k^{(r)}| < \infty \), then (11) implies

\[ \left| \lim x \right| < \frac{1}{2} \left( 1 + \sum_{r=1}^{\infty} |k^{(r)}| \right) \|x\|. \]
If \( \sum_{i=1}^{\infty} |k^{(i)}| = \infty \), then for any positive number \( M \) we have
\[ |k^{(1)}| + |k^{(2)}| + \cdots + |k^{(m)}| = K > 4M \]
for some \( m \) large enough.
Let \( \beta_i = \frac{(2M |k^{(i)}|)}{(Kk^{(i)})} \) for \( k^{(i)} \neq 0 \) and \( i \leq m \), \( \beta_i = 0 \) for \( k^{(i)} = 0 \) or \( i > m \). Construct \( x_n \) and \( z_n \) according to (2) and (8), then \( \lim_{n \to \infty} x_n = 2M \) and by (9) whenever \( n > m \), \( |z_n - z_{n+1}| < (mB)|b_n| \). So \( (z_2 - z_1) + (z_3 - z_2) + \cdots \) converges absolutely. Hence
\[ x_{2n} = (z_2 - z_1) + (z_4 - z_3) + \cdots + (z_{2n} - z_{2n-1}) \to M + \xi, \]
\[ x_{2n+1} = z_{2n+1} - x_{2n} \to M - \xi. \]

(i) \( \xi = 0 \). Then \( \lim x = M \) and \( \|x\| < 1 \).
(ii) \( \xi \neq 0 \). There is an integer \( p > m \) such that \( |b_{p+1}| + |b_{p+2}| + \cdots < 1/2 |\xi| \). Let
\[ x'_n = x_n \quad (n \leq p), \]
\[ x'_n = x_n + \xi \quad (n > p \text{ is odd}), \]
\[ x'_n = x_n - \xi \quad (n > p \text{ is even}). \]
Then \( \lim x' = M \) and \( \|x'\| < 1 \).

Hence \( \lim x \) is unbounded.

REMARK. There are cases for both possibilities. Suppose \( |b_k| = O(k^{-\alpha}), \alpha > 3/2 \). It is not difficult to verify (using the second inequality of (4)) that \( |k^{(i)}| = |b_r - b_{r+1} + \cdots| + O(r^{-2(\alpha-1)}) \). Then \( \sum_{r=1}^{\infty} |k^{(i)}| \) converges or diverges with \( \sum_{r=1}^{\infty} |b_r - b_{r+1} + \cdots| \).

REFERENCES


UNIVERSITY OF HONG KONG