BOUNDEDNESS OF LIMITS[1]

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This is an answer to a problem of A. Wilansky [2]. Suppose $\sum_1^\infty b_k$ converges absolutely. Out of each infinite sequence $x=\{x_n\}$ we may construct

$$(1)\qquad \beta_1(x) = x_1, \qquad \beta_n(x) = x_n + x_{n-1} + \sum_1^{n-1} b_k x_k \qquad (n>1).$$

Obviously $\|x\| = \text{l.u.b.}\ \{|\beta_1(x)|, |\beta_2(x)|, \cdots\}$ forms a norm over the linear space of all convergent sequences. The problem is whether the linear functional $\lim x = \lim_{n\to\infty} x_n$ is bounded in this norm.

We may solve (1) to have

$$(2)\qquad x_n = \sum_{r=1}^{n} (-1)^{n-r} g_n^{(r)} \beta_r(x)$$

where

$$(3)\qquad g_n^{(n)} = 1, \quad g_n^{(r)} = \begin{vmatrix} 1+b_r & 1 & 0 & \cdots & 0 \\ b_r & 1+b_{r+1} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_r & b_{r+1} & b_{r+2} & \cdots & 1+b_{n-1} \end{vmatrix} \qquad (r<n).$$

On expanding $g_n^{(r)}$ along the diagonal we have $g_n^{(r)} = 1 +$ linear combination of products of $b_r, \cdots, b_{n-1}$ with factors of $(1-b_r)(1-b_{r+1}) \cdots (1-b_{n-1})$ as coefficients. Hence

$$(4)\qquad |g_n^{(r)}| < B, \qquad |g_n^{(r)} - 1| < C(|b_r| + \cdots + |b_{n-1}|)$$

where B, C are independent of n and r.

By (3) we also have

$$(5)\quad g_n^{(r)} - g_n^{(r+1)} = b_r g_n^{(r+1)} - b_r \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1+b_{r+2} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & b_{r+2} & b_{r+3} & \cdots & 1+b_{n-1} \end{vmatrix},$$

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[1] Editorial note: After this paper had been accepted for publication a specific counterexample to Wilansky's conjecture was published by Hayman and Wilansky [1].

$$g_n^{(r)} - g_{n-1}^{(r)} = b_{n-1}g_{n-1}^{(r)} - b_{n-2}g_{n-2}^{(r)} + \cdots + (-1)^{n-r}b_{r+1}g_{r+1}^{(r)} + (-1)^{n-r-1}b_r. \tag{6}$$

Expanding (5) along the diagonal and arguing as in (4),

$$\left| g_n^{(r)} - g_n^{(r+1)} \right| < D\left| b_r \right| \tag{7}$$

where D is independent of n and r.

Let

$$z_1 = x_1, \qquad z_n = x_n + x_{n-1}, \qquad \beta_{n,r} = \beta_r - \beta_{r+1} + \cdots + (-1)^{n-r}\beta_n,$$

then

$$z_n = k_n^{(1)}\beta_1(x) + \cdots + k_n^{(n-1)}\beta_{n-1}(x) + \beta_n(x) \tag{8}$$

where

$$k_n^{(r)} = (-1)^{n-r}g_n^{(r)} + (-1)^{n-r-1}g_{n-1}^{(r)} = -b_r + b_{r+1}g_{r+1}^{(r)} - \cdots + (-1)^{n-r}b_{n-1}g_{n-1}^{(r)}. \tag{9}$$

Since $\beta_{r+2s+1,r} = x_{r-1} - x_{r+2s+1} - (b_r x_r + \cdots + b_{r+2s}x_{r+2s})$, we have

$$\left| \beta_{n,r} \right| < E \tag{10}$$

where E is independent of n and r. Then

$$\left| \sum_{r=s+1}^{n-1} k_n^{(r)}\beta_r(x) \right| = \left| k_n^{(s+1)}\beta_{n,s+1} + \sum_{r=s+1}^{n-2} (k_n^{(r)} + k_n^{(r+1)})\beta_{n,r} + k_n^{(n-1)}\beta_{n,n-1} \right|$$
$$\leqq E\left\{ (2C + 2D)(\left| b_{s+1} \right| + \cdots + \left| b_{n-1} \right|) + 2C\left| b_{n-1} \right| \right\}.$$

Hence, from (8),

$$\lim x = \frac{1}{2}\left[\lim_{n\to\infty} \beta_n(x) + \sum_{1}^{\infty} k^{(r)}\beta_r(x) \right] \tag{11}$$

where

$$k^{(r)} = \lim_{n\to\infty} k_n^{(r)} = \sum_{s=0}^{\infty} (-1)^{s-1} g_{r+s}^{(r)} b_{r+s}.$$

THEOREM. *The linear functional* lim *is bounded if and only if* $\sum_1^\infty \left| k^{(r)} \right| < \infty$.

PROOF. If $\sum_1^\infty \left| k^{(r)} \right| < \infty$, then (11) implies

$$\left| \lim x \right| < \frac{1}{2}\left(1 + \sum_{1}^{\infty} \left| k^{(r)} \right| \right) \|x\|.$$

If $\sum_1^\infty |k^{(r)}| = \infty$, then for any positive number M we have $|k^{(1)}|+|k^{(2)}|+\cdots+|k^{(m)}|=K>4M$ for some m large enough. Let $\beta_i=(2M|k^{(i)}|)/(Kk^{(i)})$ for $k^{(i)}\neq 0$ and $i\leqq m$, $\beta_i=0$ for $k^{(i)}=0$ or $i>m$. Construct x_n and z_n according to (2) and (8), then $\lim_{n\to\infty} z_n=2M$ and by (9) whenever $n>m$, $|z_n-z_{n+1}|<(mB)|b_n|$. So $(z_2-z_1)+(z_4-z_3)+\cdots$ converges absolutely. Hence

$$x_{2n} = (z_2 - z_1) + (z_4 - z_3) + \cdots + (z_{2n} - z_{2n-1}) \to M + \xi,$$

$$x_{2n+1} = z_{2n+1} - x_{2n} \to M - \xi.$$

(i) $\xi=0$. Then $\lim x=M$ and $\|x\|<1$.

(ii) $\xi\neq 0$. There is an integer $p>m$ such that $|b_{p+1}|+|b_{p+2}|+\cdots<1/2|\xi|$. Let

$$\begin{aligned} x_n' &= x_n && (n \leqq p),\\ x_n' &= x_n + \xi && (n > p \text{ is odd}),\\ x_n' &= x_n - \xi && (n > p \text{ is even}). \end{aligned}$$

Then $\lim x'=M$ and $\|x'\|<1$.

Hence lim is unbounded.

REMARK. There are cases for both possibilities. Suppose $|b_k|=O(k^{-\alpha})$, $\alpha>3/2$. It is not difficult to verify (using the second inequality of (4)) that $|k^{(r)}|=|b_r-b_{r+1}+\cdots|+O(r^{-2(\alpha-1)})$. Then $\sum_{r=1}^\infty |k^{(r)}|$ converges or diverges with $\sum_{r=1}^\infty |b_r-b_{r+1}+\cdots|$.

REFERENCES

1. W. Hayman and A. Wilansky, *An example in summability*, Bull. Amer. Math. Soc. **67** (1961), 554–555.

2. A. Wilansky, *An elementary inequality*, Bull. Amer. Math. Soc. **67** (1961), 355.

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