THE RECIPROCAL OF A FOURIER SERIES[1]

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Edrei and Szegö [1] have posed the following problem: *Given the Fourier coefficients of a function $G(x)$ find the Fourier coefficients of the reciprocal of the function without actually evaluating $G(x)$.* They were able to solve this problem in the case that $G(x) \geqq 0$. Unfortunately this restriction eliminates the interesting case of complex-valued functions such as those which arise in Laurent series.

In this note it is shown possible to obtain a solution without the restriction, $G(x) \geqq 0$. This is achieved by first treating the more general problem of finding the coefficients of $F(x)/G(x)$, given the coefficients of $F(x)$ and $G(x)$.

Edrei and Szegö confine attention to classical Fourier series. Their problem is treated here for arbitrary orthogonal series expansions.[2]

Let $\theta_j(x)$, $j=1, 2, \cdots$, be a sequence of bounded orthonormal functions in some region R of a space S. Thus

$$\int_R \theta_j(x)\theta_k^*(x)dx = \delta_{jk} \tag{1}$$

where the asterisk denotes the complex conjugate. The Fourier coefficients of a function $F(x) \in L$ are denoted by f_j and are given by the integral

$$f_j = \int_R F(x)\theta_j^*(x)dx, \qquad j = 1, 2, \cdots. \tag{2}$$

If $f_j = 0$ for all j it is assumed that $F(x) = 0$ almost everywhere. In other words the orthonormal sequence is closed in L.

PROBLEM 1. *Given the Fourier coefficients f_n and g_n of two functions $F(x)$ and $G(x)$ find the Fourier coefficients h_n of the function $H(x) = F(x)/G(x)$.*

A related question concerns a direct determination of $H(x)$ without first finding $F(x)$ and $G(x)$. This may be stated as

PROBLEM 2. *In terms of f_n and g_n find expansion coefficient sets $(\rho_1^{(m)}, \rho_2^{(m)}, \cdots, \rho_m^{(m)})$ for $m=1, 2, \cdots$ such that*

$$\lim_{m\to\infty} \int_R \left| H(x) - \sum_{k=1}^{m} \rho_k^{(m)}\theta_k(x) \right| dx = 0. \tag{3}$$

Received by the editors April 28, 1961 and, in revised form, December 2, 1961.

[1] Prepared under Contract DA-36-061 ORD-490, U. S. Army Research Office.

[2] This generalization was suggested to the writer by David Moskovitz.

Given a solution of Problem 2 then

$$\lim_{m\to\infty}\int_R\left[H(x)-\sum_{k=1}^{m}\rho_k^{(m)}\theta_k(x)\right]\theta_j^*(x)dx=0. \tag{4}$$

This is seen to give

$$h_j=\lim_{m\to\infty}\rho_j^{(m)} \tag{5}$$

and consequently the solution of Problem 2 also furnishes a solution of Problem 1. (In the following $\rho_j^{(m)}$ is denoted by ρ_j.)

PROBLEM 3. *The solution of Problem 2 is sought but given the "Fourier matrix"*

$$g_{jk}=\int_R G(x)\theta_j^*(x)\theta_k(x)dx \tag{6}$$

instead of the Fourier coefficients g_j. In other words find h_n in terms of f_n and g_{jk}. The following formal considerations indicate the close relationship between Problem 2 and Problem 3. Let

$$\beta_{jkr}=\int_R\theta_j^*(x)\theta_k(x)\theta_r(x)dx. \tag{7}$$

Then

$$\theta_j^*(x)\theta_k(x)=\sum_{r=1}^{\infty}\beta_{jkr}\theta_r^*(x). \tag{8}$$

This relation is multiplied by $G(x)$ and integrated termwise to yield

$$g_{jk}=\sum_{r=1}^{\infty}\beta_{jkr}g_r. \tag{9}$$

Thus if (9) is valid then a solution of Problem 3 will yield a solution of Problem 2.

THEOREM 1. *Let $G(x)\geqq 0$, $G(x)\in L$, $1/G(x)\in L$, and $F^2(x)/G(x)\in L$. Then for each integer m the equations*

$$f_j=\sum_{k=1}^{m}g_{jk}\rho_k,\qquad j=1,\cdots,m, \tag{10}$$

may be solved for $\rho_1,\rho_2,\cdots,\rho_m$ and these are expansion coefficient sets for $H(x)=F(x)/G(x)$ and this solves Problem 3.

PROOF. Since $F = HG^{1/2}G^{1/2}$ the Schwarz inequality gives

$$\left(\int |F|\,dx\right)^2 \leqq \int |H^2G|\,dx \int |G|\,dx,$$

and

$$\left(\int |H|\,dx\right)^2 \leqq \int |H^2G|\,dx \int |G^{-1}|\,dx.$$

This proves that $F \in L$ and $H \in L$. Then it is seen that the following integral exists for any choice of the constants ρ_j:

$$E = \int \left| H(x) - \sum_1^m \rho_k\theta_k(x) \right|^2 G(x)dx. \tag{11}$$

Since $G(x) \geqq 0$ this may be written in the form

$$E = \int \left| HG^{1/2} - \sum_1^m \rho_k\theta_kG^{1/2} \right|^2 dx. \tag{12}$$

This leads to consideration of the relation

$$\int U(x)\theta_j^*(x)G^{1/2}dx = 0, \qquad j = 1, 2, \cdots, \tag{13}$$

where $U(x) \in L^2$. Then $UG^{1/2} \in L$ and since the sequence θ_j is closed in L relation (13) implies $UG^{1/2} = 0$ almost everywhere. Thus $U = UG^{1/2}G^{-1/2} = 0$ almost everywhere. This proves that the sequence $\theta_jG^{1/2}$ is closed in L^2.

Closure in L^2 implies completeness in L^2 so it is possible to make E arbitrarily small. This may be accomplished by choosing, for each m, the set $(\rho_1, \cdots, \rho_m)$ which minimizes E. As is well known this optimal choice is determined by the orthogonality equations

$$0 = \int \left(H - \sum_1^m \rho_k\theta_k\right)\theta_j^*Gdx, \qquad j = 1, 2, \cdots, m. \tag{14}$$

Since $HG = F$ it is apparent that these are precisely equations (10) of Theorem 1.

By the Schwarz inequality

$$\left(\int \left| H - \sum_1^m \rho_k\theta_k \right| dx\right)^2 \leqq E \int |G^{-1}|\,dx. \tag{15}$$

The left side approaches zero if E approaches zero. This shows that the optimal choice of ρ_j leads to the satisfaction of condition (3) of Problem 2 and completes the proof.

THEOREM 2. *Suppose $F(x)\in L^2$ and $1/F(x)\in L^2$. Let*

$$\beta_{jkrs} = \int_R \theta_j^*(x)\theta_k(x)\theta_r^*(x)\theta_s(x)dx, \tag{16}$$

$$g_{jk} = \sum_1^\infty \sum_1^\infty \beta_{jkrs} f_r^* f_s. \tag{17}$$

Then the equations

$$f_j = \sum_1^m g_{jk}\rho_k, \qquad j = 1, 2, \cdots, m, \tag{18}$$

may be solved for $\rho_1, \rho_2, \cdots, \rho_m$. For $m=1, 2, \cdots$ these solutions form a set of expansion coefficients of the function $H(x)=1/F^*(x)$.

PROOF. Define $G(x)=F(x)F^*(x)$ then $H(x)=F(x)/G(x)$. Then $G(x)\geqq 0$, $G(x)\in L$, $(G(x))^{-1}\in L$, and $H^2(x)G(x)\in L$. Thus the conditions of Theorem 1 are satisfied. Substituting $G=FF^*$ in relation (6) for the Fourier matrix gives

$$g_{ik} = \int \theta_j^*\theta_k F^* F dx. \tag{19}$$

The series $\sum_1^\infty f_r^*\theta_r^*$ and $\sum_1^\infty f_s\theta_s$ converge in L^2 mean to F^* and F. Therefore it is permissible to substitute these series in (19) and to integrate termwise. This justifies (17); the proof then follows from Theorem 1.

THEOREM 3. *Let $G(x)\geqq 0$, $G(x)\in L$, $1/G(x)\in L$, and $F^2(x)/G(x)\in L$ in the interval $0\leqq x\leqq 1$. The Fourier coefficients are now defined as*

$$f_j = \int_0^1 F(x)\exp(-2\pi ijx)dx, \qquad j = 0, \pm 1, \cdots. \tag{20}$$

Then the equations

$$f_j = \sum_{-m}^m g_{j-k}\rho_k, \qquad j = -m, \cdots, m, \tag{21}$$

may be solved for $\rho_{-m}, \cdots, \rho_m$ and

$$\int_0^1 \left| F(x)/G(x) - \sum_{-m}^m \rho_k \exp(2\pi ikx) \right| dx \to 0 \text{ as } m\to\infty. \tag{22}$$

PROOF. Theorem 1 is specialized with the sequence $\theta_j(x)$ being the sequence $\exp(2\pi ijx)$ suitably reordered. It is then seen that

$$g_{jk} = g_{j-k} \tag{23}$$

and thus equation (10) becomes equation (21). The proof of relation (22) then follows from Theorem 1.

THEOREM 4. *Suppose* $F(x)\in L^2$ *and* $1/F(x)\in L^2$ *on the interval* $0\leqq x\leqq 1$. *Let* f_j *denote the Fourier coefficients of* $F(x)$ *relative to the orthonormal sequence* $\exp(2\pi ijx)$. *Let*

$$g_j = \sum_{-\infty}^{\infty} f_{j+r}f_r^*. \tag{24}$$

Then the equations

$$f^*_{-j} = \sum_{-m}^{m} g_{j-k}\alpha_k \tag{25}$$

may be solved for $\alpha_{-m}, \cdots, \alpha_m$ *and*

$$\int_0^1 \left| (F(x))^{-1} - \sum_{-m}^{m} \alpha_k \exp(2\pi ikx) \right| dx \to 0 \text{ as } m\to\infty. \tag{26}$$

PROOF. First Theorem 2 is applied to obtain $H=1/F^*$. It is seen from (16) that $\beta_{jkrs}=1$ if $s=j-k+r$ and $\beta_{jkrs}=0$ otherwise. Thus (17) gives

$$g_{jk} = g_{j-k} = \sum_{-\infty}^{\infty} f_{j-k+r}f_r^*.$$

Then (18) is

$$f_j = \sum_{-m}^{m} g_{j-k}\rho_k.$$

Taking the complex conjugate of this relation gives

$$f^*_{-j} = \sum_{-m}^{m} g^*_{k-j}\rho^*_{-k}.$$

But $g^*_{-k}=g_k$ and $\rho^*_{-k}=\alpha_k$ so this proves (25).

The questions studied in this note originated from the need for an algorithm which would give the reciprocal of a Laurent series. (In this connection it is worth noting that there is a simple algorithm for the reciprocal of a Taylor series.) An algorithm for the reciprocal of a Laurent series is stated in the following Corollary of Theorem 4.

COROLLARY. *Consider the Laurent series*

$$S(z) = \sum_{-\infty}^{\infty} a_n z^n \tag{27}$$

which is assumed to converge when the complex variable z is in the region D defined by the relation $r_1<|z|<r_2$. If $S(z)$ does not vanish in D it follows, of course, that the reciprocal $T(z)=1/S(z)$ is a Laurent series in D. Thus

$$T(z) = \sum_{-\infty}^{\infty} c_n z^n. \tag{28}$$

Let coefficients b_j be defined as

$$b_j = \sum_{-\infty}^{\infty} a_{j+r} a_r^*. \tag{29}$$

Then the equations

$$a_{-j}^* = \sum_{-m}^{m} b_{j-k} C_k^{(m)}, \qquad j = -m, \cdots, m, \tag{30}$$

may be solved for $C_{-m}^{(m)}, \cdots, C_m^{(m)}$. Let

$$T_m(z) = \sum_{-m}^{m} C_k^{(m)} z^k. \tag{31}$$

Then as $m\to\infty$ the sequence $T_m(z)$ converges uniformly to $T(z)$ in any closed region contained in D. Moreover, $C_k^{(m)}\to c_k$ as $m\to\infty$.

Proof. The function $F(x)=S(re^{2\pi ix})$ satisfies the condition of Theorem 4 for any value of r in the range $r_1<r<r_2$. The Fourier coefficients of F are given by $f_j=r^j a_j$. Defining b_j and $C_j^{(m)}$ by the relations $g_j=r^j b_j$ and $\alpha_j=r^j C_j^{(m)}$ it results that relations (24) and (25) yield relations (29) and (30). It follows that T_m converges in mean to T for $|z|=r$. Likewise T_m converges in mean for $|z|=r'$ where $r_1<r<r'<r_2$. Then by a standard theorem from complex function theory it follows that T_m converges uniformly to T for $r<|z|<r'$. This is seen to complete the proof of the Corollary.

Reference

1. A. Edrei and G. Szegö, *A note on the reciprocal of a Fourier series*, Proc. Amer. Math. Soc. 4 (1953), 323–329.

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