lows from our lemma by much the same sort of argument that produced our basic theorem.

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UNCOUNTABLY MANY NONISOMORPHIC NILPOTENT LIE ALGEBRAS[1]

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Throughout this note, L denotes a Lie algebra over the real number field R. We shall define L^i and L_i inductively. $L=L^0=L_0$, $L^i=[L^{i-1}, L^{i-1}]$, and $L_i=[L, L_{i-1}]$ for all integers $i\geqq 1$. Thus, L^i is the space of all finite sums $\sum[x, y]$, $x, y\in L^{i-1}$. Similarly, L_i is the space of all finite sums $\sum[x, y]$, $x\in L$ and $y\in L_{i-1}$. If $L^r=0$ and $L^{r-1}\neq 0$, L is said to be solvable of index r. If $L_t=0$ and $L_{t-1}\neq 0$, L is said to be nilpotent of length t.

Definition. Let F be a subfield of R. A Lie algebra L over R is said to be an F-algebra if its structure constants with respect to some basis of L lie in F.

Malcev [1] showed that for each integer $n\geqq 16$ there is a nilpotent Lie algebra of length 2 and dimension n which is not a rational algebra. The purpose of this note is to prove the following theorem which contains an improvement of Malcev's result:

Theorem. *There exist uncountably many nonisomorphic nilpotent Lie algebras of length* 2 *for any given dimension* $N\geqq 10$.

Following from the theorem we can easily show:

Corollary 1. *There exist uncountably many solvable not nilpotent Lie algebras of index* 3 *for any given dimension* $M\geqq 11$.

Received by the editors November 13, 1961.

[1] This is a portion of my thesis submitted to the University of Michigan. I am deeply grateful to Professor H. Samelson for his guidance and assistance. This work was supported by the contract AF 49(648)—104 and Lotta B. Backus scholarship.

Let E be a subfield of R, let m, n be two natural numbers, and let c_{jk}^i, $i=1, 2, \cdots, n$, $j, k=1, 2, \cdots, m$, be real numbers such that $c_{jk}^i=-c_{kj}^i$. Also let L be a Lie algebra over R defined by a basis $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ with products $[x_j, x_k]=\sum_{i=1}^n c_{jk}^i y_i$ for $j, k=1, 2, \cdots, m$, and all other products zero, so that L is nilpotent of length $\leqq 2$.

LEMMA. *If the numbers c_{jk}^i, $1\leqq i\leqq n$, $1\leqq j<k\leqq m$, are algebraically independent over E, and if $(n/2)(m^2-m)>m^2+n^2$, then L is not an E-algebra.*

PROOF. We first note that $(n/2)(m^2-m)>m^2+n^2$ implies $(1/2)(m^2-m)>n$. Any n different elements $[x_j, x_k]$, $j<k$, of L_1 are linearly independent, since the determinant formed by c_{jk}^i involved cannot be zero by the algebraic independence of all c_{jk}^i. It follows that L_1 is generated by $y_1, y_2, \cdots, y_n$, denoted by $L_1=((y_1, y_2, \cdots, y_n))$, since for any $x\in L_1$ there exist u_i, v_i such that $x=\sum_i[u_i, v_i]$ which is a linear combination of y_i's. We also note that the center of L is exactly L_1; let x be any element of the center, then $x=\sum_{j=1}^m a_j x_j+\sum_{r=1}^n b_r y_r$ and $0=[x, x_k]=\sum_{j=1}^m a_j\sum_{i=1}^n c_{jk}^i y_i$ for $k=1, 2, \cdots, m$. By linear independence of the $\{y_i\}$, we have $\sum_{j=1}^m a_j c_{jk}^i=0$ for $i=1, 2, \cdots, n$, and $k=1, 2, \cdots, m$, i.e., there are $n\cdot m$ equations and m unknowns. By the algebraic independence of all c_{jk}^i, the rank of the coefficient matrix in the system of homogeneous equations is equal to m. Hence, we have $a_1=a_2=\cdots=a_m=0$. Consequently, $x=\sum_{r=1}^n b_r y_r$ and the center is L_1, and L is of length 2.

Suppose now that L is an E-algebra with basis $(z_1, \cdots, z_m, z_{m+1}, \cdots, z_{m+n})$ and structure constants d_{jk}^i, $1\leqq i, j, k\leqq m+n$, lying in E. We can assume that $(z_1, \cdots, z_m)$ are independent modulo L_1, i.e., they span a complement C of L_1, in L. We can write $z_{m+i}=v_i+t_i$ with $v_i\in C$ and $t_i\in L_1$ for $i=1, 2, \cdots, n$. Clearly, $(z_1, \cdots, z_m, t_1, \cdots, t_n)$ is still a basis for L. We have

$$[z_i, z_j]=\sum_{r=1}^m d_{ij}^r z_r+\sum_{s=m+1}^{m+n} d_{ij}^s v_{s-m}+\sum_{s=m+1}^{m+n} d_{ij}^s t_{s-m},$$

for $1\leqq i, j\leqq m$. But since $[z_i, z_j]\in L_1$, the first two sums, which are in C, must be zero. Hence we have

$$[z_i, z_j]=\sum_{r=1}^n d_{ij}^{m+r} t_r, \quad \text{for} \quad i, j=1, 2, \cdots, m.$$

These equations describe the multiplication in L in the basis $(z_1, \cdots, z_m, t_1, \cdots, t_n)$; the structure constants are part of the structure constants for the basis $(z_1, \cdots, z_m, z_{m+1}, \cdots, z_{m+n})$.

We note that $((x_1, \cdots, x_m))$ also forms a complement of L_1 in L say C'. It follows that we can replace each z_i by an element s_i such that $s_i - z_i \in L_1$ and $s_i \in C'$. Since L_1 is the center of L, the structure constants for the basis $(s_1, \cdots, s_m, t_1, \cdots, t_n)$ are the same as for the basis $(z_1, \cdots, z_m, t_1, \cdots, t_n)$ above.

The set of vectors $\{s_1, \cdots, s_m\}$ is of course a basis for C', and, therefore, we have $s_i = \sum_{p=1}^{m} a_{ip}x_p$, $i = 1, \cdots, m$, where $A = (a_{ip})$ is a nonsingular matrix. Similarly, $t_g = \sum_{r=1}^{n} b_{gr}y_r$, $g = 1, \cdots, n$, with nonsingular matrix $B = (b_{gr})$. Substituting into $[s_i, s_j] = \sum_{u=1}^{n} d_{ij}^{m+u} t_u$, $1 \leqq i, j \leqq m$, we obtain, by linear independence,

$$\sum_p \sum_g a_{ip} a_{jg} \overset{r}{c}_{pg} = \sum_u \overset{m+u}{d_{ij}} b_{ur},$$

for fixed i, j, and r.

This means, with $\bar{a}_{ij} = (A^{-1})_{ij}$, that

$$\overset{r}{c}_{pg} = \sum_i \sum_j \sum_u \overset{m+u}{d_{ij}} b_{ur} \bar{a}_{pi} \bar{a}_{gj}.$$

These equations imply that the c_{pg}^r lie in the field $E(a_{ip}, b_{ur})$, but this field has degree of transcendency over E at most m^2+n^2 which is a contradiction. Hence, L is not an E-algebra.

The smallest dimension to which this applies is 10 with $m=6$ and $n=4$. In fact, the lemma applies to any dimension $N \geqq 10$, because when $N \geqq 10$, $N^2 - 10N + 8 > 0$ holds, implying that $n(m^2-m)/2 > m^2+n^2$ holds for $n=4$ and $m=N-4$.

Now the proof of the theorem: It is well known that there exists a set, S, of uncountably many real numbers which are algebraically independent over the rational number field Q. With $n=4$ and $m=N-4$, we divide S into disjoint subsets $(c_{jk}^i)_\alpha$ (the Greek index distinguishes the various subsets), each of which is restricted to values of j and k such that $j<k$ and $c_{jk}^i = -c_{kj}^i$. Write $(L)_\alpha = ((x_1, \cdots, x_m, y_1, \cdots, y_4))$ with products $[x_j, x_k] = \sum_{i=1}^{4} c_{jk}^i y_i$, $j, k = 1, 2, \cdots, m$, and all other products zero. There are still uncountably many such subsets $\{c_{jk}^i\}$ since each $\{c_{jk}^i\}$ is finite. Consequently, there are uncountably many such Lie algebras $(L)_\alpha$. We claim that any two $(L)_\alpha$ and $(L)_{\alpha'}$ are nonisomorphic. Since $(c_{jk}^i)_\alpha$ are algebraically independent over $Q(\{(c_{jk}^i)_{\alpha'}\})$, apply the lemma with $E = Q(\{(c_{jk}^i)_{\alpha'}\})$.

Now the proof of Corollary 1: In the proof of the theorem we have seen that for each α, $(L)_\alpha = ((x_1, \cdots, x_{N-4}, y_1, \cdots, y_4))$, with $[x_j, x_k] = \sum_{i=1}^{4} c_{jk}^i y_i$ for $j, k = 1, \cdots, N-4$ where $N \geqq 10$, and all other products zero, is a nilpotent Lie algebra of dimension N and length 2. Let $(L')_\alpha = ((x_1, \cdots, x_{N-4}, y_1, \cdots, y_4, z_\alpha))$ where the multiplications of x_j's and y_i's are defined as same as in L and $[z_\alpha, x_j] = x_j$, $[z_\alpha, y_i] = 2y_i$ for $j = 1, \cdots, N-4$; $i = 1, \cdots, 4$ and $N \geqq 10$. Then clearly, $(L)_\alpha$ is a solvable not nilpotent Lie algebra of dimension $M = N+1 \geqq 11$. Any two such Lie algebra $(L')_{\alpha'}$ and $(L')_\alpha$ are clearly nonisomorphic because by the theorem their commutators are nonisomorphic.

COROLLARY 2. *There are uncountably many nonisomorphic nonrational nilpotent Lie algebras of length* 2 *for any given dimension* $N \geqq 10$.

REMARKS. We note that the uncountability of nonisomorphic solvable Lie algebras is quite different from the case of semisimple Lie algebras where in each dimension there are only a finite number of nonisomorphic ones.

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