REPRODUCING KERNELS AND PRINCIPAL FUNCTIONS[1]

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1. **Introduction.** Using the notation and terminology of Ahlfors-Sario [3], we shall denote by Γ_h the Hilbert space of square integrable harmonic differentials on a Riemann surface W. Let γ be a 1-chain on W and let ψ_γ be the element in Γ_h with the property, $(\omega, \psi_\gamma) = \int_\gamma \omega$ for all $\omega \in \Gamma_h$. We refer to ψ_γ as a reproducing kernel for periods for Γ_h. Let ζ be a point of W and $z = x+iy$ be a local parameter near ζ such that $z(\zeta) = 0$. A Γ_h-kernel for nth derivatives at ζ is a differential $\psi_\zeta \in \Gamma_h$ which satisfies $(\omega, \psi_\zeta) = (\partial^n/\partial x^n)u(0)$ for all $\omega \in \Gamma_h$ where $\omega = du(z)$ near ζ. This kernel is uniquely determined by ζ, the uniformizer z, and the positive integer n. If in the above definitions we replace Γ_h by one of its subspaces then we shall refer to the corresponding kernels as reproducing kernels for that subspace. It is easily seen that such a kernel is the orthogonal projection of a Γ_h-kernel.

The existence of these kernels is well known (see [1; 3]). The possibility of expressing them explicitly in terms of principal harmonic functions (the existence of which has been proved constructively by Sario [5]) was first investigated by Weill [6]. In this paper the kernels for the spaces Γ_h, Γ_{hse}, Γ_{he}, Γ_{hm}, Γ_{h0}, $\Gamma_{he} \cap \Gamma_{hse}^*$, $\Gamma_{h0} \cap \Gamma_{hse}^*$, Γ_a, Γ_{ase}, and some spaces associated with a regular partition of the ideal boundary of the surface are found in terms of principal functions.

§§9 and 10 deal with some applications of these results. A characterization of Ahlfors' class of quasi-rational functions is given in terms of principal functions. A Riemann-Roch type theorem is proved for arbitrary Riemann surfaces which reduces to a theorem of Royden [4] in case the surface is of class O_{KD}.

2. **Reproducing kernels.** We shall apply the general method of normal operators [3] to the following situation. Let h be a harmonic function defined in a boundary neighborhood of a Riemann surface W' and assume that h has zero flux along the ideal boundary of W'. The principal operator L_0 maps h into a function p_{0h} which is harmonic on W' and is characterized up to an additive constant by the

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property that the normal derivative of $p_{0h}-h$ vanishes identically along the ideal boundary of W'. When W' is the complement of a finite subset $\{\zeta_1, \cdots, \zeta_n\}$ of a Riemann surface W then $p_{0h}-h$ has a removable singularity at each ζ_i. We say that it has L_0 behavior at the ideal boundary of W. Let P be a regular partition of this ideal boundary [**3**, p. 165]. The $(P)L_1$ principal operator associates to h a function p_{Ph}, harmonic on W' with the singularity h at each ζ_i. Furthermore, $p_{Ph}-h$ is constant on each ideal boundary component of W determined by P and has zero flux over each P-dividing cycle.

Let I and Q denote, respectively, the identity and the canonical partitions of the ideal boundary of W in the sense of [**3**]. In terms of a uniformizer z near $\zeta \in W$ such that $z(\zeta)=0$, we let $s=\operatorname{Re} z^{-n}$ and $t=\operatorname{Im} z^{-n}$ for a positive integer n, and define $s\equiv t\equiv 0$ outside of a compact subset of W. The differential $dp_{Is}+dp_{It}^*$ has a removable singularity at ζ and is an element of Γ_h.

THEOREM 1. *The differential $dp_{Is}+dp_{It}^*$ reproduces for the space Γ_h. Explicitly, we have*

$$(\omega, dp_{Is} + dp_{It}^*) = -\frac{2\pi}{(n-1)!}\frac{\partial^n}{\partial x^n} u(0)$$

where $\omega=du(z)$ near ζ.

3. Before proving the theorem we shall first establish a lemma. A subregion $\Omega \subset W$ is called canonical if it is relatively compact, its boundary consists of a finite number of analytic Jordan curves, and each component of $W-\Omega$ is noncompact and bounded by a single contour. For a first order differential ω let $\int_\beta \omega$ indicate the limit as $\Omega \to W$ of integrals taken along the boundaries of exhausting canonical subregions Ω.

LEMMA. *Let p be the $(I)L_1$ principal function for any singularity. Then $\int_\beta p\omega=0$ for all $\omega \in \Gamma_h$.*

If $\Omega \subset W$ is a canonical subregion with boundary $\partial\Omega$ then $\left|\int_{\partial\Omega} p\omega\right| = \left|\int_{\partial\Omega}(p-p_\Omega)\omega\right|$, where p_Ω is the $(I)L_1$ principal function for the surface Ω which has the same singularity as that of p. By Stokes' theorem the right side of this equation is the inner product $(dp-dp_\Omega, \bar{\omega}^*)_\Omega$ taken over the region Ω. By Schwarz's inequality this inner product is dominated by $\|dp-dp_\Omega\|_\Omega \cdot \|\omega\|_\Omega$. We therefore obtain $\left|\int_\beta p\omega\right| \leqq \lim_{\Omega\to W} \|dp-dp_\Omega\|_\Omega \cdot \|\omega\|_W$. The uniform convergence $p_\Omega \to p$ on compact subsets of W implies that $\|dp-dp_\Omega\|_\Omega \to 0$.

For the proof of Theorem 1 we may assume that ω is real. Let Δ be a relatively compact neighborhood of ζ which is mapped by z onto a disk $\{|z|<r\}$ and let $\partial\Delta=\alpha$. In order to simplify the notation we shall temporarily denote the functions p_{I_s} and p_{I_t} by p and q respectively. In Δ let $\omega=du(z)$ and $p=\operatorname{Re} z^{-n}+v(z)$ where u and v are harmonic in $\{|z|<r\}$. We then obtain

$$(\omega, dp+dq^*)=\iint_W \omega(dp^*-dq)$$
$$=\iint_{W-\Delta} \omega dp^*-\omega dq+\iint_\Delta \omega(dp^*-dq).$$

The integral over $W-\Delta$ is the limit as $\Omega\to W$ of integrals over $\Omega-\Delta$. We may apply Stokes' theorem to these integrals and, after simplification, we find

$$(\omega, dp+dq^*)=\int_\beta (p\omega^*+q\omega)-\int_\alpha (p\omega^*+p^*\omega).$$

The integral along β vanishes by the lemma. Hence

$$(\omega, dp+dq^*)=-\int_{|z|=r}(\operatorname{Re} z^{-n}+v)du^*+(\operatorname{Im} z^{-n}+v^*)du$$
$$=\operatorname{Re}\int(u^*-iu)d(z^{-n})-\int vdu^*-udv^*$$
$$=\operatorname{Re} ni\int \frac{u+iu^*}{z^{n+1}}dz$$
$$=\frac{2\pi}{(n-1)!}\frac{\partial^n}{\partial x^n}u(0).$$

4. We now seek an expression for the reproducing differential for periods. Let Δ be a parametric disk on W and γ a 1-simplex contained in Δ. In terms of the parameter z which maps Δ onto the unit disk we define a singularity function $\sigma=\log|(z-\zeta_2)/(z-\zeta_1)|$, where $\partial\gamma=\zeta_2-\zeta_1$. We set $\sigma\equiv 0$ outside of a compact set. The corresponding $(P)L_1$ principal function is $p_{P\sigma}$. Let τ be the singularity function $\tau=\arg(z-\zeta_2)/(z-\zeta_1)$ in $\Delta-\gamma$ and $\tau\equiv 0$ near the ideal boundary of W. On the surface $W-\gamma$ we choose the normal operator which is composed of $(P)L_1$ for a boundary neighborhood of W and of the Dirichlet operator for $\Delta-\gamma$. This Dirichlet operator maps a continuous function on $\partial\Delta$ into the restriction to $\Delta-\gamma$ of the harmonic function in Δ with these boundary values. The direct sum of these operators yields a function $p_{P\tau}$ harmonic on $W-\gamma$. The differential $dp_{P\tau}$ can be ex-

tended harmonically to all of $W-\{\zeta_1, \zeta_2\}$. We shall continue to denote the extension by $dp_{P\tau}$, even though it is not exact. If γ is an arbitrary 1-chain it is homologous to a finite sum $\sum n_i\gamma_i$ where each γ_i is a 1-simplex contained in a parametric disk and each n_i is an integer. We extend the definitions of $dp_{P\sigma}$ and $dp_{P\tau}$ to arbitrary γ by letting $dp_{P\sigma}=\sum n_i dp_{P\sigma_i}$ and similarly for $dp_{P\tau}$. That these differentials are well defined will follow from Theorem 2.

THEOREM 2. *The differential $dp_{I\sigma}+dp_{I\tau}^*$ reproduces for Γ_h. Specifically, if σ and τ correspond to a* 1*-chain γ then*

$$(\omega, dp_{I\sigma} + dp_{I\tau}^*) = 2\pi \int_\gamma \omega$$

for all $\omega \in \Gamma_h$.

Because of the linearity it suffices to prove Theorem 2 for the case that γ is a 1-simplex contained in a parametric disk Δ: $\{|z|<1\}$ and $\partial\gamma=\zeta_2-\zeta_1$. We shall shorten the notation and write p and q for $p_{I\sigma}$ and $p_{I\tau}$ respectively. As in the proof of Theorem 1 it can be shown that

$$(\omega, dp + dq^*) = -\int_\alpha (p\omega^* + p^*\omega),$$

where $\partial\Delta=\alpha$. In Δ let $\omega=du$ and $p=\log|(z-\zeta_2)/(z-\zeta_1)|+v$, where u and v are harmonic. Let α_1 and α_2 be disjoint circles in Δ with centers at $z(\zeta_1)$ and $z(\zeta_2)$ respectively. After applying Green's formula one obtains

$$(\omega, dp + dq^*) = \int_{\alpha_2} ud\log|z-\zeta_2|^* - \int_{\alpha_1} ud\log|z-\zeta_1|^*.$$

By the mean value formula the last two integrals reduce to $2\pi(u(\zeta_2)-u(\zeta_1))$. We have shown that $(\omega, dp+dq^*)=2\pi\int_{\zeta_1}^{\zeta_2}du=2\pi\int_\alpha\omega$. This completes the proof of Theorem 2.

5. For a regular partition P of the ideal boundary of W the spaces $(P)\Gamma_{hm}$ and $(P)\Gamma_{hse}$ were introduced in [3], and a proof of the decomposition $\Gamma_h=(P)\Gamma_{hm}\dot{+}(P)\Gamma_{hse}^*$ was sketched. It follows that $\Gamma_h=\Gamma_{he}^*\dot{+}(P)\Gamma_{hm}\dot{+}(P)\Gamma_{hse}^*\cap\Gamma_{h0}$ is an orthogonal direct sum. Consider the identity

$$(1) \quad -(dp_{Is} + dp_{It}^*) = (dp_{0t} - dp_{It})^* + (dp_{Ps} - dp_{Is}) - (dp_{0t}^* + dp_{Ps}),$$

where the zero subscript refers to the L_0 principal function. The first term on the right side of (1) is evidently in Γ_{he}^*. The second term is the limit as $\Omega \to W$ of $dp_{P_s\Omega} - dp_{I_s\Omega}$, where the subscript Ω indicates a principal function for the subsurface Ω. These approximating differentials belong to the space $(P)\Gamma_{hm}(\Omega)$ of harmonic differentials du on Ω with u constant on each set of contours of the boundary of Ω which belong to the same part in the induced partition. We may conclude that the limit differential is in $(P)\Gamma_{hm}$. The last term on the right side of (1) is a limit of elements from $\Gamma_{h0}(\Omega)$. By Theorem V.14C of [3], we see that it is in Γ_{h0}. Its conjugate has vanishing periods along any cycle which is dividing relative to the partition P. Hence it is in $(P)\Gamma^*_{hse} \cap \Gamma_{h0}$. Thus (1) represents the reproducing kernel for Γ_h, except for a multiplicative constant, as the sum of the kernels for Γ^*_{he}, $(P)\Gamma_{hm}$, and $(P)\Gamma^*_{hse} \cap \Gamma_{h0}$. The reasoning is also valid for the kernels corresponding to a 1-chain γ. This proves the following two corollaries.

COROLLARY 1. *The differentials*

$$\frac{(n-1)!}{2\pi}(dp_{P_s} - dp_{I_s})$$

and

$$\frac{1}{2\pi}(dp_{I_\sigma} - dp_{P_\sigma})$$

are the reproducing kernels for the space $(P)\Gamma_{hm}$.

COROLLARY 2. *The differentials*

$$\frac{-(n-1)!}{2\pi}(dp_{P_s} + d\overset{*}{p}_{0t})$$

and

$$\frac{1}{2\pi}(dp_{P_\sigma} + d\overset{*}{p}_{0\tau})$$

are the reproducing kernels for $(P)\Gamma^*_{hse} \cap \Gamma_{h0}$.

Since $(I)\Gamma^*_{hse} \cap \Gamma_{h0} = \Gamma_{h0}$, Corollary 2 contains the following special case.

COROLLARY 3. *The differentials*

$$\frac{-(n-1)!}{2\pi}(dp_{I_s} + d\overset{*}{p}_{0t})$$

and

$$\frac{1}{2\pi}(dp_{I\sigma} + dp_{0\tau}^*)$$

are the reproducing kernels for Γ_{h0}.

REMARKS. In order to indicate explicitly the relation between the differentials in (1) and the classical domain functions we denote by $N(z, \zeta)$ and $G(z, \zeta)$ the Neumanns' and Green's functions for W. In case W is parabolic let $G(z, \zeta) = G(z, z_0; \zeta, \zeta_0)$ be the fundamental potential [4]. In terms of the complex notation

$$d' = \frac{\partial}{\partial z} dz, \qquad d'' = \frac{\partial}{\partial \bar{z}} d\bar{z}, \qquad d^C = i(d'' - d')$$

(so that $d^C f = df^*$), we have

$$dp_{Is} + dp_{It}^* = \frac{4}{(n-1)!} \operatorname{Re} d'' \frac{\partial^n G}{\partial \zeta^n},$$

$$(dp_{0t} - dp_{It})^* = \frac{2}{(n-1)!} \operatorname{Im} d^C \frac{\partial^n}{\partial \zeta^n}(N - G),$$

$$dp_{I\sigma} + dp_{I\tau}^* = 4 \operatorname{Re} \int_{\zeta \in \gamma} d_z'' d_\zeta' G.$$

Theorems 1 and 2 can also be derived from the fact that $d''(\partial G/\partial \zeta)$ reproduces for Γ_a. If W is the interior of a compact bordered surface then by classical theory there is a potential $G_P(z, z_0; \zeta, \zeta_0)$ which as a function of z is constant and has zero flux along each P-boundary component. In this case we have

$$dp_{Ps} - dp_{Is} = \frac{2}{(n-1)!} \operatorname{Re} d \frac{\partial^n}{\partial \zeta^n}(G_P - G),$$

$$dp_{0t}^* + dp_{Ps} = \frac{2}{(n-1)!} \operatorname{Re}(dG_P - id^C N).$$

6. The space Γ_h has the orthogonal decomposition

$$\Gamma_h = (P)\Gamma_{hm} \dotplus \Gamma_{h0}^* \dotplus (P)\Gamma_{hse}^* \cap \Gamma_{he}.$$

Since we are in possession of reproducing kernels for $(P)\Gamma_{hm}$ and Γ_{h0}^* (see the proof of Corollary 1) we obtain the next result immediately.

COROLLARY 4. *The differentials*

$$\frac{(n-1)!}{2\pi}(dp_{0s} - dp_{Ps})$$

and

$$\frac{1}{2\pi}(dp_{P\sigma} - dp_{0\sigma})$$

are the reproducing kernels for $(P)\Gamma^*_{hse} \cap \Gamma_{he}$.

The reproducing kernel for Γ_{he} may be derived from Corollary 4.

COROLLARY 5. *The differentials*

$$\frac{(n-1)!}{2\pi}(dp_{0s} - dp_{Is})$$

and

$$\frac{1}{2\pi}(dp_{I\sigma} - dp_{0\sigma})$$

are the reproducing kernels for Γ_{he}.

7. The kernel for $(P)\Gamma_{hse}$ can be found from the identity $-(dp_{Is} + dp^*_{It}) = (dp^*_{Pt} - dp^*_{It}) - (dp_{Is} + dp^*_{Pt})$ and the orthogonal decomposition $\Gamma_h = (P)\Gamma^*_{hm} \dot{+} (P)\Gamma_{hse}$.

COROLLARY 6. *The differentials*

$$\frac{-(n-1)!}{2\pi}(dp_{Is} + dp^*_{Pt})$$

and

$$\frac{1}{2\pi}(dp_{I\sigma} + dp^*_{P\tau})$$

are the reproducing kernels for $(P)\Gamma_{hse}$.

8. From the orthogonal direct sum,

$$\Gamma_h = (P)\Gamma_{hm} \dot{+} (P)\Gamma^*_{hm} + (P)\Gamma_{ase} + (P)\bar{\Gamma}_{ase},$$

we can project the Γ_h-kernels into $(P)\Gamma_{ase}$.

COROLLARY 7. *The differentials*

$$\frac{-(n-1)!}{4\pi}(dp_{Ps} + dp^*_{Pt} + i(dp^*_{Ps} - dp_{Pt}))$$

and

$$\frac{1}{4\pi}\left(dp_{P_\sigma} + d\overset{*}{p}_{P_\tau} + i(d\overset{*}{p}_{P_\sigma} - dp_{P_\tau})\right)$$

are the reproducing kernels for $(P)\Gamma_{ase}$.

The kernels for Γ_a and Γ_{ase} may be found immediately from Corollary 7.

REMARKS. It is an open question whether results analogous to those above can be obtained for the subspaces Γ_S and Γ_{aS}. Investigations in this area might lead to new normal operators.

9. The results of §§1–8 indicate a connection between principal functions and Ahlfors' generalization of Abel's theorem [**2**, **3**]. Recall that a differential ω on W, harmonic except for harmonic poles, is called *distinguished* if

(i) ω^* is semiexact outside of some compact subset of W,

(ii) there exist differentials $\omega_{hm} \in \Gamma_{hm}$ and $\omega_{e0} \in \Gamma_{e0}^1$ such that $\omega = \omega_{hm} + \omega_{e0}$ in a boundary neighborhood of W.

LEMMA. *Let* γ *be a* 1*-chain on* W *and* δ *be a* 1*-cycle. Let* dp_{P_σ} *and* dp_{P_τ} *be differentials associated with* γ (see §4). *Then*

$$\int_\delta (dp_{P_\sigma} + i\,dp_{P_\tau}) = 2\pi i(\delta \times \gamma). \tag{2}$$

PROOF. Because (2) is linear in γ we may assume that γ is a simplex contained in a parametric disk Δ: $\{|z| < 1\}$. Since $dp_{P_\sigma} + i\,dp_{P_\tau}$ has no periods along cycles in $W - \Delta$ we may even assume $\delta \subset \bar{\Delta}$. Let $\partial\gamma = \zeta_2 - \zeta_1$. Then $dp_{P_\sigma} + i\,dp_{P_\tau} = d \log (z - \zeta_2)/(z - \zeta_1) + du$ in Δ, where u is harmonic. The proof reduces to establishing

$$\int_\delta d \log \frac{z - \zeta_2}{z - \zeta_1} = 2\pi i(\delta \times \gamma). \tag{3}$$

Since $\delta \times \gamma$ is the number of times γ crosses δ from left to right, (3) follows from the argument principle.

With the help of the above lemma it can be seen that to each distinguished differential ω there corresponds a differential $\lambda(P, \omega)$ with the singularities and periods of ω and which, in a boundary neighborhood of W, is the differential of a function whose real and imaginary parts have $(P)L_1$-behavior.

THEOREM 3. *Let* ω *be a distinguished differential. Then* $\lambda(Q, \omega) = \omega$.

PROOF. Because of the uniqueness theorem for distinguished differentials it suffices to prove merely that $\lambda(Q, \omega)$ satisfies conditions (i) and (ii) above. The first condition is obvious.

The differential $\omega_{hm}=\lambda(Q, \omega)-\lambda(I, \omega)$ is in Γ_{hm}. Let Ω be a canonical subregion of W such that ω is regular and exact in $W-\Omega$. Choose another canonical region $\Omega'\supset\bar{\Omega}$. By standard techniques one can construct an exact differential ω_{e0} in Γ^1 with the property,

$$\omega_{e0} = \begin{cases} 0 \text{ in } \Omega, \\ \lambda(I, \omega) \text{ in } W - \Omega'. \end{cases}$$

That $\omega_{e0}\in\Gamma_{e0}$ follows from the fact that an $(I)L_1$ principal function for W is the limit, uniformly on compacta, of $(I)L_1$ principal functions for bordered exhausting canonical subregions. The equality $\lambda(Q, \omega)=\omega_{hm}+\omega_{e0}$ is valid in $W-\Omega'$ and hence $\lambda(Q, \omega)$ is distinguished.

From the above considerations one sees that *a meromorphic function f is quasi-rational if and only if* $\log f$ *has a single-valued branch outside of a compact subset of W, the real and imaginary parts of which are $(Q)L_1$ principal functions.*

REMARK. There is a natural interpretation for "quasi-rational with respect to a regular partition P." The corresponding generalized Abel's theorem is obtained by replacing Γ_{ase} by $(P)\Gamma_{ase}$.

10. **Riemann-Roch type theorems.** The classical theorem of Riemann-Roch has been extended to surfaces of class O_{KD} by Royden [4]. By the methods of this paper we obtain a similar theorem for an arbitrary Riemann surface.

Let D be a divisor on a Riemann surface W and let $D=B-A$ where A and B are disjoint integral divisors. Suppose that $A=\sum m_jA_j$ and $B=\sum n_kB_k$, where the A_j and B_k are points of W and the m_j and n_k are positive integers. For each j and k let Δ_j and Δ_k' be parametric disks with centers at A_j and B_k respectively. We assume that these disks are mutually disjoint. Let $\Delta=\bigcup\Delta_j$ and $\Delta'=\bigcup\Delta_k'$.

Let $S=s+it$ be a meromorphic function in Δ' and a multiple of the divisor $-B$. Define $S\equiv 0$ in a boundary neighborhood of W and form the differential $dF_S=dp_{Ps}+idp_{Pt}$. Let $\mathfrak{B}$ denote the complex vector space consisting of all such dF_S. The dimension of $\mathfrak{B}$ is equal to the degree of B, i.e., dim $\mathfrak{B}=\deg B=\sum n_k$.

In the decomposition $dF_S=\phi_S+\bar{\psi}_S$, where $\bar{\psi}_S=\frac{1}{2}(dF_S-idF_S^*)$, the differential ϕ_S is meromorphic and ψ_S is in $(P)\Gamma_{ase}$. According to Corollary 7, ψ_S has certain reproducing properties. The following lemma is a generalization of that fact. For a divisor E let $(P)\Gamma_{ase}[E]$ denote the vector space of meromorphic differentials which are multiples of E, square integrable near the ideal boundary of W, and which have vanishing periods along each P-dividing cycle outside of some compact set.

LEMMA. *Let* $\alpha \in (P)\Gamma_{ase}[-A]$. *Then*

$$(\alpha, \psi_S)_{W-\Delta} = i \int_{\partial\Delta'} S\alpha + i \int_{\partial\Delta} F_S \alpha. \tag{4}$$

The inner product is to be understood as a Cauchy limit. That is, it is the limit of inner products taken over $W-\Delta-\Delta'$ as the radii of the parametric disks of Δ' tend to zero. Note that

$$(\alpha, \psi_S)_{W-\Delta-\Delta'} = (dF_S, \bar{\alpha})_{W-\Delta-\Delta'} = -i \lim_{\Omega \to W} \int_{\partial(\Omega-\Delta-\Delta')} F_S \alpha.$$

A modification of the proof of the lemma in §3 shows that $\int_{\partial\Omega} F_S \alpha \to 0$ as $\Omega \to W$. In Δ' we have $F_S = S + u$ where S is meromorphic and u is harmonic. When the disks about the B_k shrink to points we obtain (4).

We shall make use of the following algebraic facts [**3**, p. 325]. Let U and V be vector spaces over a field K. A bilinear mapping $T: U \times V \to K$ is called a *pairing* of U and V. The *left kernel* U_0 is the space $\{u \in U: T(u, V) = 0\}$, and the *right kernel* V_0 is the space $\{v \in V: T(U, v) = 0\}$. If one of the quotient spaces U/U_0 or V/V_0 is finite dimensional then there is an isomorphism $U/U_0 \cong V/V_0$.

Consider the pairing $T: \mathfrak{B} \times (P)\Gamma_{ase}[-A] \to \mathbf{C}$ defined by $T(dF_S, \alpha) = \int_{\partial\Delta'} S\alpha$.

Suppose dF_S is in the left kernel of this pairing. From (4) we have $(\alpha, \psi_S)_{W-\Delta} = i\int_{\partial\Delta} F_S \alpha$ for all $\alpha \in (P)\Gamma_{ase}[-A]$. We may replace α by ψ_S and conclude that $\|\psi_S\|_W = 0$. Hence $\psi_S = 0$ and dF_S is meromorphic. We also find that

$$\int_{\partial\Delta} F_S \alpha = 0. \tag{5}$$

For appropriate choices of α we conclude from (5) that the additive constant in F_S can be chosen so that F_S is a multiple of the divisor $-D$. Conversely, if dF_S is meromorphic then $\psi_S = 0$ and $T(dF_S, \alpha) = \int_{\partial\Delta'} S\alpha = -\int_{\partial\Delta} F_S \alpha$. The differential α has a pole of order at most m_j at A_j, and $F_S^{(k)}(A_j) = 0$ $(k = 1, \cdots, m_j - 1)$ if F_S is a multiple of A. Hence $\int_{\partial\Delta} F_S \alpha = 0$ by Cauchy's integral formula. Thus dF_S is in the left kernel $\mathfrak{B}_0$ if and only if the function F_S is meromorphic and can be normalized so as to be a multiple of $-D$.

A differential $\alpha \in (P)\Gamma_{ase}[-A]$ is in the right kernel if and only if

$\int_{\partial\Delta'} S\alpha = 0$ for all S which are multiples of $-B$. Convenient choices for S show that α must be a multiple of B, i.e., $\alpha \in (P)\Gamma_{ase}[D]$.

Since $\mathfrak{B}$ is finite dimensional we have

$$\mathfrak{B}/\mathfrak{B}_0 \cong (P)\Gamma_{ase}[-A]/(P)\Gamma_{ase}[D],$$

or

$$\dim \mathfrak{B}_0 = \dim \mathfrak{B} - \dim(P)\Gamma_{ase}[-A]/(P)\Gamma_{ase}[D]. \tag{6}$$

Let $(P)M$ be the complex vector space of meromorphic functions on W whose real and imaginary parts have $(P)L_1$ behavior near the ideal boundary. Denote by $(P)M[E]$, where E is a divisor, the subspace of $(P)M$ consisting of functions which are multiples of E. The homomorphism d: $(P)M[-D] \rightarrow \mathfrak{B}_0$: $F_S \rightarrow dF_S$ is surjective. The kernel is $\mathbf{C}$ if deg $A = 0$, and it is zero if deg $A \neq 0$. From (6) we obtain

$$\begin{aligned} \dim(P)M[-D] = \deg B + 1 - \min(1, \deg A) \\ - \dim(P)\Gamma_{ase}[-A]/(P)\Gamma_{ase}[D]. \end{aligned} \tag{7}$$

THEOREM 4. *Let A and B be disjoint integral divisors on a Riemann surface W and let $D = B - A$. Then* (7) *is valid for any regular partition P of the ideal boundary of W.*

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