W^*-ALGEBRAS WITH A SINGLE GENERATOR

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In [4] the author set forth a complete set of unitary invariants for a certain class of operators on Hilbert space. The operators considered were exactly those operators which generate a finite W^*-algebra of type I in the terminology of [2]. One immediately wants to know some nontrivial examples of such operators, and Brown provided several examples in [1]. (Nontrivial here means non-normal operator on infinite-dimensional Hilbert space.) It is the purpose of this note to show that there exists an abundance of such operators, in the sense of the following theorem.

THEOREM. *If R is any W^*-algebra of operators acting on a separable Hilbert space, and R is of type* I, *then there exists an operator $A \in R$ which generates R (in the sense that R is the smallest W^*-algebra containing A).*

We first prove the following lemma.

LEMMA. *If n is any cardinal number satisfying $1 \leqq n \leqq \aleph_0$, and $\mathcal{H}$ is any n-dimensional Hilbert space, then there is an operator A on $\mathcal{H}$ such that the W^*-algebra generated by A is $\mathcal{L}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$.*

PROOF. Whether n is finite or infinite, it clearly suffices to exhibit an operator A which has no nontrivial reducing subspace. In case n is finite, take A to be any operator with n distinct eigenvalues and with the property that no two eigenvectors corresponding to different eigenvalues are orthogonal. In case $n = \aleph_0$, choose an orthonormal basis $\{x_i\}$, $i = 1, 2, \cdots$, for $\mathcal{H}$ and define A by setting $Ax_i = x_{i+1}$, $i = 1, 2, \cdots$. That A has no nontrivial reducing subspace is proved on page 356 of [5].

We now prove the theorem, using von Neumann's result in [3] that any abelian W^*-algebra on a separable Hilbert space has a single Hermitian generator and results of Dixmier in [2].

PROOF OF THE THEOREM. One knows (see [1] for example) that R is a direct sum $\sum_{n \in N} \oplus R_n$ where each R_n is an n-homogeneous algebra and N is some set of cardinal numbers bounded above by $\aleph_0$. We suppose first that the theorem is known for homogeneous algebras, and return to the proof of this case later. For each $n \in N$, let B_n generate R_n, and arrange it so that the B_n are uniformly bounded in norm.

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Then $B=\sum_{n\in N}\oplus B_n\in R$. Let C be a generator for the center of R. Then one sees immediately that the W^*-algebra generated by the pair (B, C) contains each homogeneous algebra R_n, and therefore must be R. We now obtain a single operator generating R as follows. Write $B=H+iK$, H and K Hermitian. Let $A_1=A_1^*$ generate the same abelian W^*-algebra as the pair (H, C) and let $A_2=A_2^*$ generate the same algebra as (K, C). Then take $A=A_1+iA_2$.

We return now to deal with the homogeneous case. Let R be an n-homogeneous W^*-algebra ($n\leqq\aleph_0$), and let I, the unit of R, be the identity operator on the separable Hilbert space $\mathcal{H}$. Then I can be written as $I=\sum_{i=1}^n E_i$, where the E_i are mutually orthogonal, equivalent, abelian projections in R. Let $\mathcal{H}_1=E_1(\mathcal{H})$, let $\mathcal{H}_2$ be a Hilbert space of dimension n, and let $\mathcal{K}=\mathcal{H}_1\otimes\mathcal{H}_2$ (the tensor product of $\mathcal{H}_1$ with $\mathcal{H}_2$). It follows from Proposition 5, page 27 of [2], that R is unitarily isomorphic to the (tensor product) W^*-algebra $R_1=E_1RE_1\otimes\mathcal{L}(\mathcal{H}_2)$ of operators acting on the Hilbert space $\mathcal{K}$, and thus it suffices to obtain a single generator for R_1. From von Neumann's result in [3] we obtain a single generator C for the abelian algebra E_1RE_1, and from the lemma we obtain a single generator B for $\mathcal{L}(\mathcal{H}_2)$. Let $G=C\otimes I_{\mathcal{H}_2}$, and let $D=I_{\mathcal{H}_1}\otimes B$. It follows from Proposition 6, page 28 of [2], that the pair (G, D) generates R_1, and the argument is completed as above.

Remarks. (1) It is immediate from Exercise 3, page 119 of [2], that one cannot hope to extend this result to algebras of type I on nonseparable spaces.

(2) Is it the case that every W^*-algebra (regardless of type) acting on a separable space has a single generator?

Bibliography

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