A THEOREM ON INDUCED REPRESENTATIONS

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In [1], we proved a criterion for the disjointness of two induced representations U^L and U^M of a Lie group G, where L and M are finite-dimensional unitary representations of compact subgroups H and K, respectively, of G. The purpose of this paper is to improve this theorem by getting a stronger conclusion, while dropping the conditions that G be a Lie group and H be compact, and that L and M be finite-dimensional. Moreover, the restriction on K is weakened to read: G has arbitrarily small neighborhoods of the identity invariant under the adjoint action of K on G. Finally, the proof given below is fairly elementary, while the proof in [1] is quite involved.

Notations and conventions: Let $\mathfrak{U}$ be a topological vector space. $C(G, \mathfrak{U})$ will denote the space of continuous functions from G to $\mathfrak{U}$, equipped with the topology of uniform convergence on compact subsets of G while $C_0(G, \mathfrak{U})$ will denote the space of those $f \in C(G, \mathfrak{U})$ with compact support. If $\mathfrak{U}$ is omitted it is understood that $\mathfrak{U} = \mathbf{C}$. If $\mathfrak{U}_1$ is another topological vector space, $\mathfrak{L}(\mathfrak{U}, \mathfrak{U}_1)$ will denote the space of continuous linear maps from $\mathfrak{U}$ into $\mathfrak{U}_1$ equipped with the topology of bounded convergence. All integrations are with respect to right Haar measure. For any locally compact group G, δ_G will denote its modular function. If $f, g \in C_0(G)$, $f \circ g$ will denote the convolution of f and g, and f^* is defined by $f^*(x) = \delta_G(x)^{-1} f(x^{-1})^-$. If L and M are representations of G, $R(L, M)$ will denote the space of intertwining operators for L and M (see [3]), while $I(L, M)$ will denote the dimension of $R(L, M)$. For the definition of induced representation used below, see [1]. Finally, for any function f on the group G and any $x \in G$, f_x and f^x are defined by $f_x(y) = f(x^{-1}y)$ and $f^x(y) = f(yx)$.

THEOREM. *Let H and K be closed subgroups of the locally compact group G. Let L (resp. M) be a unitary representation of H (resp. K) on the Hilbert space $\mathfrak{V}$ (resp. $\mathfrak{W}$). Suppose that G has arbitrarily small neighborhoods of the identity invariant under the adjoint action of K on G. Let $\mathfrak{M}$ be the subspace of those $S \in C(G, \mathfrak{L}(\mathfrak{V}, \mathfrak{W}))$ such that*

$$S(\xi^{-1} x \eta) = \delta_H(\xi)^{-1/2} \delta_G(\xi)^{1/2} M_\eta^{-1} S(x) L_\xi$$

for all $\xi \in H$, $\eta \in K$, and $x \in G$. Then

$$I(U^L, U^M) \leqq \dim \mathfrak{M},$$

provided $\dim \mathfrak{M}$ *is finite.*

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PROOF. We recall from [1] the bilinear map ϵ_H defined on $C_0(G)\times\mathcal{V}$ with values in the Hilbert space of U^L:

$$\epsilon_H(f, v)(x) = \int_H \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}f(\xi x)L_\xi^{-1}v\,d\xi$$

for $f\in C_0(G)$, $v\in\mathcal{V}$. ϵ_K is similarly defined from $C_0(G)\times\mathcal{W}$ into the space of U^M. Given $f, g\in C_0(G)$, $v\in\mathcal{V}$, and $w\in\mathcal{W}$, we define the linear functional $\Phi(f, g, v, w)$ on $R(U^L, U^M)$ by setting $\Phi(f, g, v, w)(A) = (A\epsilon_H(f, v), \epsilon_K(g, w))$ and the linear functional $\Psi(f, g, v, w)$ on $\mathfrak{M}$ by setting

$$\Psi(f, g, v, w)(S) = \int_G (f\circ g^*)(x)(S(x)v, w)dx.$$

Φ and Ψ are linear in f and v are are conjugate-linear in g and w.

Let $\mathcal{S}$ (resp. $\mathcal{T}$) be the linear space of functionals spanned by the Φ's (resp. Ψ's). It is obvious that $\mathcal{T}$ separates $\mathfrak{M}$. That $\mathcal{S}$ separates $R(U^L, U^M)$ follows from [1, Lemma 2]. We shall show that there is a linear map of $\mathcal{T}$ onto $\mathcal{S}$ which sends each $\Psi(f, g, v, w)$ onto $\Phi(f, g, v, w)$, and this will accomplish the proof.

According to [1, Lemma 2], there is a constant C_Q for each compact set Q in G such that

$$|\Phi(f, g, v, w)(A)| \leqq C_Q\|f\|_\infty\|g\|_\infty\|v\|\,\|w\|\,\|A\|$$

whenever f and g have their supports in Q. From this it follows easily that for each $h\in C_0(G)$, $A\in R(U^L, U^M)$, and $x\in G$ there is an operator $T_h(A)(x)\in\mathcal{L}(\mathcal{V}, \mathcal{W})$ such that $(T_h(A)(x)v, w) = \delta_G(x)^{-1}\Phi(h_x, h, v, w)(A)$ for all $v\in\mathcal{V}$ and $w\in\mathcal{W}$ and that $T_h\in\mathcal{L}(R(U^L, U^M), C(G, \mathcal{L}(\mathcal{V}, \mathcal{W})))$.

Now the standard proof of the complete regularity of G (cf. [4, pp. 28–30]) may be modified to show that for any neighborhood N of e in G, there is a nonzero positive continuous function h whose support is in N such that $h(\eta x) = h(x\eta)$ for all $x\in G$ and $\eta\in K$: one need merely require all neighborhoods of e mentioned therein to be invariant under the adjoint action of K on G. Let h be such a K-invariant function. For every $\eta\in K$, $\delta_G(\eta)\int_G h(x)dx = \int_G h(\eta^{-1}x)dx = \int_G h(x\eta^{-1})dx = \int_G h(x)dx$. Since $\int_G h(x)dx > 0$, it follows that $\delta_G|K = 1$. Similarly $\delta_K = 1$.

If h is a K-invariant function in $C_0(G)$, then $T_h(A)\in\mathfrak{M}$ for all $A\in R(U^L, U^M)$. In fact, $\epsilon_H(f_{\xi^{-1}}, v) = \delta_H(\xi)^{-1/2}\delta_G(\xi)^{-1/2}\epsilon_H(f, L_\xi v)$ and $\epsilon_H(f^x, v) = U_x^L\epsilon_H(f, v)$ for all $f\in C_0(G)$, $v\in\mathcal{V}$, $\xi\in H$, and $x\in G$ (cf. the proof of Theorem 3 in [1]). Similar statements hold for ϵ_K. Also

$h_\eta = h^{\eta^{-1}}$ for $\eta \in K$. Therefore

$$\begin{aligned}(T_h(A)(\xi^{-1}x\eta)v, w) &= \delta_G(\xi^{-1}x\eta)^{-1}\Phi(h_{\xi^{-1}x\eta}, h, v, w)(A)\\ &= \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}\delta_G(x)^{-1}\Phi(h_x^{\eta^{-1}}, h, L_\xi v, w)(A)\\ &= \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}\delta_G(x)^{-1}\Phi(h_x, h^\eta, L_\xi v, w)(A)\\ &= \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}\delta_G(x)^{-1}\Phi(h_x, h, L_\xi v, M_\eta w)(A)\\ &= \delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}(M_\eta^{-1}T_h(A)(x)L_\xi v, w)\end{aligned}$$

for all $\xi \in H$, $\eta \in K$, $x \in G$, $v \in \mathcal{V}$, and $w \in \mathcal{W}$, as claimed.

We next assert that $\Psi(f, g, v, w) \circ T_h = \Phi(f, h \circ h^* \circ g, v, w)$ for $f, g, h \in C_0(G)$, $v \in \mathcal{V}$, $w \in \mathcal{W}$, where h is K-invariant. To see this, note that $f \circ g = \int g(x) f^{x^{-1}} dx = \int f(x^{-1}) g_{x^{-1}} dx$, where the integrals on the right are strong integrals in the uniform topology. Using the continuity mentioned above of Φ in its arguments, we deduce that

$$\begin{aligned}\Psi(f, g, v, w)(T_h(A)) &= \int (f \circ g^*)(x^{-1})\Phi(h_{x^{-1}}, h, v, w)(A)dx\\ &= \Phi\left(\int (f \circ g^*)(x^{-1})h_{x^{-1}}dx, h, v, w\right)(A)\\ &= \Phi(f \circ g^* \circ h, h, v, w)(A)\\ &= \Phi\left(\int (g^* \circ h)(x) f^{x^{-1}} dx, h, v, w\right)(A)\\ &= \int (g^* \circ h)(x)\Phi(f^{x^{-1}}, h, v, w)(A)dx\\ &= \int (g^* \circ h)(x)\Phi(f, h^x, v, w)(A)dx\\ &= \Phi\left(f, \int [(g^* \circ h)(x)]^- h^x dx, v, w\right)(A)\\ &= \Phi(f, h \circ (g^* \circ h)^*, v, w)(A),\end{aligned}$$

as claimed.

Finally, let λ_{fgvw} be a set of complex numbers indexed by $C_0(G) \times C_0(G) \times \mathcal{V} \times \mathcal{W}$, all but a finite number of which vanish, such that $\sum \lambda_{fgvw} \Psi(f, g, v, w) = 0$. Then $\sum \lambda_{fgvw} \Phi(f, h \circ h^* \circ g, v, w) = 0$ for all K-invariant $h \in C_0(G)$. Letting $h \circ h^*$ approach the δ-measure, we see that $\sum \lambda_{fgvw} \Phi(f, g, v, w) = 0$. Therefore the linear map of $\mathfrak{J}$ onto $\mathfrak{S}$ mentioned at the beginning of this proof exists, and our proof is complete.

For $x \in G$, we define a representation M^x of xKx^{-1} by setting $M^x_\eta = M_{x^{-1}\eta x}$ for all $\eta \in xKx^{-1}$. It is easy to see that

$$I(\delta_H^{-1/2}\delta_G^{1/2}L \mid H \cap xKx^{-1},\ M^x \mid H \cap xKx^{-1})$$

depends only on L, M, and the $H{:}K$ double coset D to which x belongs (cf. [3]). We shall denote this number by $I(L, M, D)$. In what follows, $\mathfrak{D}$ will denote the set of $H{:}K$ double cosets. We retain the hypotheses of the Theorem.

COROLLARY 1. *Let $\mathfrak{D}_0$ be the set of open $D \in \mathfrak{D}$. Suppose that* $\bigcup\{D : D \in \mathfrak{D}_0\}$ *is dense in G. Then $I(U^L, U^M) \leqq \sum\{I(L, M, D) : D \in \mathfrak{D}_0\}$, provided this sum is finite.*

(A simple category argument shows that the denseness hypothesis is satisfied if H and K are denumerable at infinity and $\mathfrak{D}$ is denumerable. Cf. [2, Lemma 3; 1].)

PROOF. To each $D \in \mathfrak{D}$, assign an element $x_D \in D$. Let

$$R = \prod \left\{ R(\delta_H^{-1/2}\delta_G^{1/2}L \mid H \cap x_D K x_D^{-1},\ M^{x_D} \mid H \cap x_D K x_D^{-1}) : D \in \mathfrak{D}_0 \right\}.$$

For each $S \in \mathfrak{M}$, let $\Lambda(S)$ be the function from $\mathfrak{D}_0$ to $\mathfrak{L}(\mathfrak{V}, \mathfrak{W})$ defined by $\Lambda(S)(D) = S(x_D)$, $D \in \mathfrak{D}_0$. Λ is linear. It is also 1-1, since $\Lambda(S) = 0$ implies that $S \mid D = 0$ if $D \in \mathfrak{D}_0$ so that S vanishes on a dense subset of G. Finally, $\mathrm{Im}(\Lambda) \subseteq R$. In fact, let $\xi \in H \cap x_D K x_D^{-1}$. Then $x_D^{-1}\xi x_D \in K$ and $S(x_D) = S(\xi^{-1}x_D(x_D^{-1}\xi x_D)) = \delta_H^{-1/2}(\xi)\,\delta_G(\xi)^{1/2}\, M_{x_D^{-1}\xi x_D^{-1}} S(x_D)\, L_\xi$, whence our assertion. The corollary is an immediate consequence of the Theorem and the properties of Λ.

COROLLARY 2. *Let $\mathfrak{D}_n$ be the set of $D \in \mathfrak{D}$ for which $I(L, M, D) = 0$. Suppose that* $\bigcup\{D : D \in \mathfrak{D}_n\}$ *is dense in G. Then $I(U^L, U^M) = 0$.*

PROOF. Let $S \in \mathfrak{M}$. The methods used in proving Corollary 1 show that $S(x_D) = 0$ for $D \in \mathfrak{D}_n$, whence S vanishes on a dense subset of G and we have our result.

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