ON PRODUCT AND BUNDLE NEIGHBORHOODS

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If a nice space X is embedded in a euclidean space, it may fail to have product neighborhoods; i.e., neighborhoods which are products of X with a ball. However, if the euclidean space is a hyperplane of a higher-dimensional euclidean space one can sometimes guarantee the existence of product neighborhoods in the big euclidean space. For example, it follows from a lemma due to Klee [1] that if X is a k-ball in $R^n \subset R^{n+k}$, then X has a neighborhood homeomorphic with $X \times B^n$ in R^{n+k} (where B^n denotes an n-ball). We obtain two results along these lines. Theorem 1 gives circumstances in which we get product neighborhoods and Theorem 2 yields ball-bundle neighborhoods. Theorem 2 has been used for smoothing combinatorial manifolds [3].

DEFINITION. We say that a space X has a *local multiplication* into a space Z if there exists a neighborhood N of the diagonal Δ of $X \times X$ and a map $\phi: N \to Z$ such that:

(i) $\phi(\Delta) = z_0 \in Z$,

(ii) $\phi | N_x$ is one-to-one, and $\phi(N_x)$ contains a fixed neighborhood W of z_0, for all x in X. (N_x = pairs in N with first coordinate x.)

EXAMPLE 1. Any topological group G has a local multiplication into itself. Take $N = G \times G$ and define $\phi(g, h) = gh^{-1}$.

EXAMPLE 2. If X is a k-dimensional parallelizable manifold, then X has a local multiplication into R^k. We take a Riemannian metric for X and choose N so that if $(x, y) \in N$, then there is a unique geodesic from x to y. Let $\tau(x, y)$ denote the vector tangent to this geodesic at x and having length equal to the length of the geodesic. Let c be a cross section of the k-frame bundle over X. (c exists since X is parallelizable.) We define $\phi(x, y)$ to be the point in R^k with coordinates equal to the dot products of $\tau(x, y)$ with the vectors of $c(x)$. It is easy to verify that ϕ is a local multiplication.

A space S is said to have the *neighborhood extension property* if for any closed subset B of a separable metric space Y and any map $f: B \to S$, there exists an extension of f to some neighborhood of B.

THEOREM 1. *Let X be a compact space which has the neighborhood extension property and has a local multiplication ϕ into Z. Let $\alpha: X \to H$*

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be an embedding of X into a locally compact separable metric group H. Then for any sufficiently small compact neighborhood U of the identity e of H, there exists a product neighborhood $X\times U$ of $\alpha(X)$ in $H\times Z$.

PROOF. We will consider that α embeds X into $H\times z_0$ and show that α can be extended to a homeomorphism ψ of $X\times U$ into $H\times Z$. Here $z_0=\phi(\Delta)$.

Extend $\alpha^{-1}: \alpha(X)\rightarrow X$ to a map β of a neighborhood V of $\alpha(X)$ into X (V is a neighborhood in H). Choose a compact neighborhood U of e in H such that $U\cdot\alpha(X)\subset V$. Let $\bar{x}=\alpha(x)$ and define

$$\psi(x, h) = (h\bar{x}, \phi\{\beta(h\bar{x}), x\}).$$

One easily checks the following properties of ψ.

(1) $\psi(x, e)=(\bar{x}, z_0)=\alpha(x)$,

(2) ψ is one-to-one and onto a neighborhood.

(3) ψ is continuous.

Since $X\times U$ is compact, ψ is a homeomorphism and the theorem is proved.

COROLLARY 1. *If G is a group with the neighborhood extension property and G is embedded in H, then G has small product neighborhoods $G\times U$ in $H\times G$.*

COROLLARY 2. *If X is a parallelizable closed k-manifold in R^n, then X has a product neighborhood $X\times B^n$ in R^{n+k}.*

For example, this is the case if X is a closed orientable 3-manifold or is a compact Lie group. Either Corollary 1 or 2 shows that a simple closed curve S^1 in R^n has a product neighborhood $S^1\times B^n$ in R^{n+1}. For $k>1$, the fact that $S^k\subset R^n$ has a product neighborhood in R^{n+k} follows from Klee's lemma and Stalling's unknotting theorem [4]. In general, we would like to show that a k-manifold X in R^n has an n-ball-bundle neighborhood in R^{n+k}. We cannot do this, but can get ball-bundle neighborhoods if we are willing to raise the dimension of the embedding space. The next theorem shows how this is done.

Let X be a compact space. We assume the following about X which will be automatically true if X is a smooth manifold: (1) the diagonal Δ of $X\times X$ has a k-ball bundle neighborhood U in $X\times X$. Precisely, we assume that there exists a k-plane bundle $\tau=\tau^k$ over X ($=\Delta$) and a homeomorphism g of U onto the vectors of length less than or equal to one (for some metric in τ) such that $g(x, y)\in\tau_x$, $(x, y)\in U$, τ_x the fibre over $x\in X$.[2] We identify U with τ_1. For X a

[2] $g(x, x)=0\in\tau_x$.

smooth manifold the normal bundle of Δ (which is the same as the tangent bundle of X) gives such a neighborhood.

Since X is compact there exists a vector bundle ν^m over X such that $\tau^k \oplus \nu^m$ is a trivial bundle. (See (2.19) and (2.20) of [2].) Since $\tau^k \oplus \nu^m$ is trivial, we have a fibre-preserving homeomorphism $\phi: \tau^k \oplus \nu^m \to X \times R^{k+m}$, and for $x \in X$ we denote its restriction to the fibre $\tau_x \oplus \nu_x$ by ϕ_x.

THEOREM 2. *Suppose X is compact, satisfies* (1), *and has the neighborhood extension property. If $\alpha: X \to R^n$ is any embedding of X in R^n, then $\alpha(X)$ has a ball-bundle neighborhood in $R^n \times R^{k+m}$.*

PROOF. Let 0^n be the product n-plane bundle over X, and consider that α embeds the zero cross section of $0^n \oplus \nu^m$. We will prove that there exists $\epsilon > 0$ and an embedding

$$\psi: 0_\epsilon^n \oplus \nu^m \to R^n \times R^{k+m}$$

which extends α.

Let $\beta: N \to X$ be an extension of α^{-1} to a neighborhood N of $\alpha(X)$. If $(x, h) \in X \times R^n$ and $\|h\| < \epsilon$, we can consider $(x, h) \in 0_\epsilon^n$. Let $\bar{x}$ denote $\alpha(X)$ and choose ϵ small enough so that, for $\|h\| < \epsilon$, $\bar{x} + h \in N$ and $(\beta(\bar{x}+h), x) \in U$.

For $x, y \in X$ we define a map $f_{xy}: \nu_x \to \nu_y$ to be the composition

$$\nu_x \xrightarrow{\text{incl}} \tau_x \oplus \nu_x \xrightarrow{\phi_x} R^{k+m} \xrightarrow{\phi_y^{-1}} \tau_y \oplus \nu_y \xrightarrow{\text{proj.}} \nu_y.$$

We note that f_{yx} is linear and if (y, x) is in a sufficiently small neighborhood of Δ in $X \times X$, then f_{yx} is an isomorphism. We assume ϵ is small enough so that $(\beta(\bar{x}+h), x)$ is in such a neighborhood for $\|h\| < \epsilon$.

For notational convenience let $\sigma = \bar{x} + h$ and $\rho = \beta(\bar{x}+h)$. For $v \in \nu_x$ we define

$$\psi((x, h) + v) = (\sigma, \phi\{(\rho, x) + f_{\rho x}(v)\}).$$

Then ψ maps $0_\epsilon^n \oplus \nu^m$ into $R^n \times R^{k+m}$, and ψ is clearly continuous. It remains to check that ψ is one-to-one.

Suppose $\psi((x_1, h_1) + v_1) = \psi((x_2, h_2) + v_2)$. Then $\sigma_1 = \sigma_2$ so $\rho_1 = \rho_2$ and hence both (ρ_1, x_1) and (ρ_2, x_2) are in the fibre τ_{ρ_1}. Similarly, both $f_{\rho_1 x_1}(v_1)$ and $f_{\rho_2 x_2}(v_2)$ are in the fiber ν_{ρ_1}. Now ϕ is an isomorphism and these fibers are disjoint, so we must have $(\rho_1, x_1) = (\rho_2, x_2)$ so that $x_1 = x_2$ and $h_1 = h_2$. Also $f_{\rho_1 x_1}$ is one-to-one so that $v_1 = v_2$ and the theorem is proved.

REMARK 1. Theorem 2 yields Corollary 2 to Theorem 1 as a special case, because if X is parallelizable then τ is trivial and ν is not needed. Hence ψ embeds $X\times B^n$ into $R^n\times R^k$. More generally

COROLLARY 1. *If the diagonal Δ of $X\times X$ has a k-ball bundle neighborhood τ_1 such that τ is stably trivial (i.e., τ plus a trivial bundle is trivial), then ψ embeds $X\times B^{n+1}$ in $R^n\times R^{k+1}$.*

REMARK 2. The proof of Theorem 2 applies to an embedding α of X into any smooth manifold V^n, if we replace o^n by the tangent bundle τ' of M restricted to X. Then ψ becomes an embedding of $(\tau'|X)_\epsilon\oplus\nu^m\rightarrow V^n\times R^{k+m}$ which extends α.

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