

ON RINGS OF WHICH ANY ONE-SIDED QUOTIENT RINGS ARE TWO-SIDED

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0. Introduction and preliminaries. Following R. E. Johnson [1] we assume in this paper that any rings we shall be concerned with satisfy either one or both of the following conditions:

(J_l) If the right annihilator of a left ideal A is nonzero, then there exists a nonzero left ideal B such that $A \cap B = 0$.

(J_r) = the right left symmetry of (J_l).¹

We say that a ring is a J_l -ring, a J_r -ring or a J -ring if it satisfies (J_l), (J_r) or both of them.

A module A is called an essential extension of a submodule B if $B \cap C \neq 0$ for every nonzero submodule C of A . A module is said to be injective if it is a direct summand of every extension module. It is well known that every module M has a maximal essential extension \bar{M} . \bar{M} is injective, and is unique to within an isomorphism over M .

Let S be a J_l -ring. Then we can define the multiplication in the maximal essential extension \bar{S} of the left S -module S such a way that (i) \bar{S} forms a ring and (ii) the multiplication coincides, on $S \times \bar{S}$, with the scalar multiplication. This ring is unique up to an isomorphism over S , and is denoted by \bar{S}_l . As is known, \bar{S}_l is regular (in the sense of von Neumann), and is left self injective, that is, injective as a left module over itself. An extension ring T of a J_l -ring S is called a left quotient ring of S if the left S -module T is an essential extension of the left S -module S . It is also known that every left quotient ring of S is isomorphic, over S , to a subring of \bar{S}_l . Thus, \bar{S}_l is the maximal left quotient ring of S .

We define similarly a right quotient ring and the maximal right quotient ring \bar{S}_r of a J_r -ring S .

For any J -ring S it is easily seen that the following conditions are equivalent:

(i) There exists an extension ring T of S with the properties that (a) it is regular (both left and right) self injective, and (b) every nonzero one-sided S -submodule of T has a nonzero intersection with S .

(ii) Every left quotient ring of S is a right quotient ring of S , and every right quotient ring of S is a left quotient ring of S .

In this case any maximal left quotient ring of S and any maximal

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¹ In the terminology of Johnson [1] (J_l) (J_r) means that the left (right) singular ideal of the ring is zero.

right quotient ring of S are mutually isomorphic over S . We denote this fact by writing $\bar{S}_l = \bar{S}_r$.

The main theorem (Theorem 3.3) in this paper states that for any J -ring S we have $\bar{S}_l = \bar{S}_r$ if and only if S satisfies the converse of (J_l) and (J_r) , that is, the following two conditions:

(K_l) If $A \cap B = 0$, $B \neq 0$, for left ideals A and B , then the right annihilator of A is nonzero;

(K_r) = the right left symmetry of (K_l) .

1. Strongly regular rings. A ring is called strongly regular if for any element x there exists an element y such that $x^2y = x$. As is well known, in this case xy is central idempotent and $xy = yx$. Every one-sided ideal of a strongly regular ring is two-sided. A regular ring is strongly regular if and only if it is of index 1, that is, it contains no nonzero nilpotent elements.

A left ideal A of a ring is said to be closed if there are no left ideals B such that $A \neq B$ and B is an essential extension of A . The set of all closed left ideals of a J_l -ring S forms a complete complemented modular lattice, which is denoted by $L(S)$. If a ring T is a left quotient ring of a J_l -ring S , then T is also a J_l -ring, and $L(T)$ is isomorphic to $L(S)$ by the correspondence $A (\in L(T)) \rightarrow A \cap S$. A left ideal of \bar{S}_l is closed if and only if it is principal. Thus, $L(\bar{S}_l)$ is the lattice of all principal left ideals of \bar{S}_l . Let $A, B \in L(S)$. Then $A \cap B$ is the intersection of A and B , while $A \cup B$ is the maximal essential extension of $A + B$.

We can define similarly the notion of a closed right ideal. $R(S)$ denotes the lattice of all closed right ideals of a J_r -ring S . $R(\bar{S}_r)$ is the lattice of principal right ideals of \bar{S}_r .

THEOREM 1.1. *Let S be a regular ring. Then the following conditions are equivalent:*

- (i) S is strongly regular.
- (ii) $L(S)$ is distributive.
- (iii) The lattice L of principal left ideals of S is distributive.

PROOF. (i) \Rightarrow (ii). Let $A, B \in L(S)$. Then $AB \subset A \cap B = (A \cap B)^2 \subset AB$, and so $AB = A \cap B$. Thus, for any $P, Q, R \in L(S)$ we have $P \cap (Q + R) = P(Q + R) = PQ + PR = (P \cap Q) + (P \cap R)$. Since $L(S)$ is complemented, $P \cap (Q \cup R) = ((P \cap Q) \cup (P \cap R)) \cup C$ and $((P \cap Q) \cup (P \cap R)) \cap C = 0$ for some $C \in L(S)$. Then $((P \cap Q) + (P \cap R)) \cap C = 0$ and $P \cap (Q + R) \cap C = 0$, hence $P \cap (Q \cup R) \cap C = 0$, which means that $C = 0$ and $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$.

(ii) \Rightarrow (iii) follows immediately since L is a sublattice of $L(S)$.

(iii) \Rightarrow (i). Let $Se = Sf$, $e = e^2$ and $f = f^2$. Then $S(1 - e) = S(1 - f)$ by (iii). Thus $e = f$. This shows that every idempotent is central. There-

fore $xyx = x$ means $x^2y = x$, as desired. In case $S \ni 1$, it follows from the above argument tSt is strongly regular for any $t = t^2$. Let $S \ni x$, and let a, b, c be idempotents such that $Sx = Sa, xS = bS$ and $aS + bS = cS$. Then $x \in tSt, t = t^2$ for $t = a + c - ac$. Therefore S is strongly regular.

COROLLARY 1.2. *If S is strongly regular, then so is \bar{S}_l .*

In fact, $L(S)$ is distributive by assumption. Since $L(S) \simeq L(\bar{S}_l), L(\bar{S}_l)$ also is distributive, and hence \bar{S}_l is strongly regular.

COROLLARY 1.3. *Let S be a J_1 -ring. Then \bar{S}_l is strongly regular if and only if $L(S)$ is distributive.*

In fact, $L(\bar{S}_l)$ is distributive if and only if so is $L(S)$.

THEOREM 1.4. *Let S be a J -ring. If S has a strongly regular left quotient ring P and a strongly regular right quotient ring Q , then $\bar{S}_l = \bar{S}_r$.*

PROOF. Let $0 \neq p \in P$. Then $ap = b \neq 0$ for some $a, b \in S$, and $a^2q = a$ for some $q \in Q$. Since $b \neq 0, 0 \neq b^2 = bap$ and $ba \neq 0$. Hence $0 \neq (ba)^2 = b(ab)a$ and $0 \neq ab = a^2qb = a(aq)b = a(aq)^2b = a^2(qaq)b = a^2(aq^2)b$, whence $aq^2b \neq 0$. Thus, $(aq^2b)c = d \neq 0$ for some $c, d \in S$. $abc = (a^2q)bc = a(aq)bc = a(aq)^2bc = a^2(qaqb)c = a^2(aq^2b)c = a^2d$, and so $a^2pc = a(ap)c = a^2d$. Hence $a^2(pc - d) = 0$ and $((pc - d)a^2)^2 = 0$, which implies that $(pc - d)a^2 = 0$ and $p(ca^2) = da^2$. Now $0 \neq d = aq^2bc = (aq)(qbc) = (aq)^3(qbc) = (aq)(qbc)(aq)^2 = (aq^2bc)a^2q^2 = da^2q^2$, and hence $da^2 \neq 0$. This shows that P is a right quotient ring of S . Thus we have proved that every strongly regular left quotient ring of S is a right quotient ring of S provided S has at least one strongly regular right quotient ring. It is not too hard to see that \bar{S}_l is the maximal left quotient ring of P . Hence \bar{S}_l is strongly regular by Corollary 1.2. Therefore \bar{S}_l is a right quotient ring of S . This implies that every left quotient ring of S is a right quotient ring of S . Similarly we can show that every right quotient ring of S is a left quotient ring of S . Thus, $\bar{S}_l = \bar{S}_r$, completing the proof.

2. **The condition (K).** For any subset Z of an extension ring of a ring S we denote by $l(S, Z)$ the set of all $x \in S$ such that $xZ = 0$. Similarly $r(S, Z)$ denotes the right annihilator, in S , of Z .

LEMMA 2.1. *Let S be a J_1 -ring, and let $A \in L(\bar{S}_l)$. Then $r(S, A \cap S) = r(S, A)$.*

PROOF. \supset is obvious. Let $(A \cap S)x = 0, x \in S$, and set $B = l(\bar{S}_l, x)$. Since $B \in L(\bar{S}_l)$ and $B \cap S \supset A \cap S$, we have $B \supset A$ by the isomorphism $L(S) \simeq L(\bar{S}_l)$. Hence $Ax = 0$ and $x \in r(S, A)$. Therefore $r(S, A \cap S) \subset r(S, A)$.

THEOREM 2.2. *Let S be a J_1 -ring. Then the following conditions are equivalent:*

(K_1) *If $A \cap B = 0$, $B \neq 0$ for two left ideals A and B , then the right annihilator of A is nonzero.*

(K'_1) *Every closed left ideal is an annihilator.*

(K''_1) *Every nonzero right ideal of \bar{S}_1 has a nonzero intersection with S .*

PROOF. $(K_1) \Rightarrow (K''_1)$. Suppose that $S \cap D = 0$ for some nonzero right ideal D of \bar{S}_1 . Let $0 \neq e = e^2 \in D$. By Lemma 2.1 $r(S, \bar{S}_1(1-e) \cap S) = r(S, \bar{S}_1(1-e)) = r(\bar{S}_1, \bar{S}_1(1-e)) \cap S = e\bar{S}_1 \cap S \subset D \cap S = 0$. Since $(\bar{S}_1(1-e) \cap S) \cap (\bar{S}_1 e \cap S) = 0$ and $\bar{S}_1 e \cap S \neq 0$, this contradicts (K_1) .

$(K''_1) \Rightarrow (K'_1)$. Let A be a closed left ideal, and set $l(S, r(S, A)) = B$. If $A \neq B$, there is a left ideal $C \neq 0$ such that $B \supset C$ and $A \cap C = 0$. Let D be a maximal left ideal such that $D \supset A$ and $D \cap C = 0$. Then D is closed, and $D = \bar{S}_1 e \cap S$ for some $e = e^2 \neq 1$. By $(K''_1) (1-e)\bar{S}_1 \cap S \neq 0$. Let $0 \neq x \in (1-e)\bar{S}_1 \cap S$. Then $Dx = 0$ and $Ax = 0$, hence $Bx = 0$ and $Cx = 0$. Thus $(D \oplus C)x = 0$, and $(D \oplus C) \cap E = 0$ for some left ideal $E \neq 0$. Hence $C \cap (D \oplus E) = 0$, which contradicts the maximality of D . Therefore $A = B$.

$(K'_1) \Rightarrow (K_1)$. Let $A \cap B = 0$ and $B \neq 0$ for left ideals A, B of S . Let C be a maximal left ideal such that $C \supset A$ and $C \cap B = 0$. Then C is closed, and hence is an annihilator by (K'_1) . If $r(S, A) = 0$, then $r(S, C) = 0$ and $C = S$, which means that $B = 0$, a contradiction. Thus $r(S, A) \neq 0$, as desired.

COROLLARY 2.3. \bar{S}_1 satisfies (K_1) for any J_1 -ring S .

In fact, \bar{S}_1 satisfies (K''_1) because \bar{S}_1 is the maximal left quotient ring of itself.²

COROLLARY 2.4. *If a J -ring S satisfies (K_1) and (K_r) , then $L(S)$ is dually isomorphic to $R(S)$ by the annihilator relation.*

This is evident from (K'_1) and its symmetry.³

COROLLARY 2.5. *Let S be a J_1 -ring, and T a left quotient ring of S .*

(i) *If S satisfies (K_1) , then so does T .*

(ii) *If T is a right quotient ring of S , and if T satisfies (K_1) , then S also satisfies it.*

In fact, \bar{S}_1 is the maximal left quotient ring of T . Let A be a nonzero right ideal of \bar{S}_1 . If S satisfies (K_1) , $A \cap S \neq 0$ by (K''_1) . Hence

² Of course, (K_1) itself also follows immediately from the injectivity of \bar{S}_1 .

³ It should be noted that every annihilator left (right) ideal of a J_1 - (J_r-) ring is closed.

$A \cap T \neq 0$, and T also satisfies (K_1') . Conversely, if T is a right quotient ring of S , $A \cap T \neq 0$ implies that $A \cap S \neq 0$. This proves (ii).

LEMMA 2.6. *Let S be a J_r -ring, and let $S = A \oplus B$, A and B being ideals. If S satisfies (K_1) , then so does A .*

PROOF. Let P, Q be left ideals of A such that $P \cap Q = 0, Q \neq 0$. Then $(P \oplus B) \cap Q = 0$. Hence $(P \oplus B)(a + b) = 0, a + b \neq 0$ for some $a \in A, b \in B$. Therefore $Sb = 0$ and $b = 0$, whence $a \neq 0$. Since $Pa = 0$, this proves that A satisfies (K_1) .

3. The main theorem. A right ideal A of S is called large if $A \cap B \neq 0$ for every right ideal $B \neq 0$.

LEMMA 3.1. *Let S be a J_r -ring, and T an extension ring. Denote by U the set of all elements x of T such that $\{y | y \in S, xy \in S\}$ is a large right ideal of S . If every nonzero left ideal of T has a nonzero intersection with S , then U is a right quotient ring of S .*

PROOF. It is easily seen that U forms a subring of T . Let V be the set of all elements $y \in T$ such that $r(S, y)$ is large. Then V is a left ideal of T , and $V \cap S = 0$ since S is a J_r -ring. Hence $V = 0$ by the assumption. It follows from this that U is a right quotient ring of S , as desired.

We proved in [3] the following

THEOREM 3.2. *Every regular left self injective ring S is decomposed into the direct sum of two ideals A and B in such a way that A is strongly regular, and B is generated by idempotents.*

In fact, by [3, Theorem 4] $S = A \oplus B$, where A is strongly regular and B does not contain any nonzero strongly regular ideals. B is generated by idempotents by [3, Theorem 2].

By virtue of this theorem we obtain

THEOREM 3.3. *Let S be a J -ring. Then $\bar{S}_l = \bar{S}_r$, if and only if S satisfies (K_l) and (K_r) .*

PROOF. Let $\bar{S}_l = \bar{S}_r$. By Corollary 2.3 \bar{S}_l satisfies (K_l) . Hence S also satisfies (K_l) by (ii) of Corollary 2.5. Similarly S satisfies (K_r) , proving the only if part of the theorem. To see the if part let U be the set of all elements x of \bar{S}_l such that $\{y | y \in S, xy \in S\}$ is a large right ideal of S . By Lemma 3.1 U is a right quotient ring of S . Let $\bar{S}_l \ni e = e^2$. Then $(\bar{S}_l e \cap S) \cap (\bar{S}_l(1 - e) \cap S) = 0$. By Corollary 2.4 $L(S)$ is dually isomorphic to $R(S)$ and the correspondence is given by the annihilator relation. Since $\bar{S}_l e \cap S, \bar{S}_l(1 - e) \cap S \in L(S)$, we have $r(S, \bar{S}_l e \cap S) \cup r(S, \bar{S}_l(1 - e) \cap S) = S$ in $R(S)$. By Lemma 2.1

$r(S, \bar{S}_i e \cap S) = r(S, \bar{S}_i e) = r(\bar{S}_i, \bar{S}_i e) \cap S = (1-e)\bar{S}_i \cap S$, and similarly $r(S, \bar{S}_i(1-e) \cap S) = e\bar{S}_i \cap S$. Therefore $((1-e)\bar{S}_i \cap S) \cup (e\bar{S}_i \cap S) = S$. This implies that $((1-e)\bar{S}_i \cap S) + (e\bar{S}_i \cap S)$ is a large right ideal of S . Evidently $e(((1-e)\bar{S}_i \cap S) + (e\bar{S}_i \cap S)) = e\bar{S}_i \cap S \subset S$. It follows from this that $e \in U$. Therefore we have proved that every idempotent in \bar{S}_i is contained in U . By Theorem 3.2 there exists a central idempotent f of \bar{S}_i such that $\bar{S}_i f$ is strongly regular, and $\bar{S}_i(1-f)$ is generated by idempotents. Since $f \in U$, $U = Uf \oplus U(1-f)$. As is easily seen $\bar{S}_i f$ is the maximal left quotient ring of Uf . Since $\bar{S}_i f$ is strongly regular, $L(Uf)$ is distributive by Corollary 1.3. Now by (i) of Corollary 2.5 U satisfies (K_i) and (K_r) , and hence Uf also satisfies them by Lemma 2.6, whence $L(Uf)$ and $R(Uf)$ are dually isomorphic by Corollary 2.4. Thus, $R(Uf)$ is distributive, and so the maximal right quotient ring of Uf is strongly regular by Corollary 1.3. Therefore $\bar{S}_i f$ is a right quotient ring of Uf by Theorem 1.4. On the other hand, since $\bar{S}_i(1-f)$ is generated by idempotents, and since U contains every idempotent in \bar{S}_i , it follows that $\bar{S}_i(1-f) = U(1-f)$. Hence $\bar{S}_i = \bar{S}_i f \oplus \bar{S}_i(1-f)$ is a right quotient ring of $U = Uf \oplus U(1-f)$. Since U is a right quotient ring of S , we see that \bar{S}_i is a right quotient ring of S . We can prove similarly that \bar{S}_r is a left quotient ring of S . Thus, we have $\bar{S}_i = \bar{S}_r$, completing the proof.

4. WR-rings and continuous rings. A ring is called a semisimple I -ring if every one-sided ideal contains a nonzero idempotent. By [2, (4, 10)] every semisimple I -ring is a J -ring. A ring is said to be of bounded index if there is an integer n such that $x^n = 0$ for every nilpotent element x . A semisimple I -ring is WR (= weakly reducible) if and only if every nonzero ideal contains a nonzero ideal of bounded index by [4, Theorem 9]. In [2, Theorem 5] we proved that $\bar{S}_i = \bar{S}_r$ for any semisimple WR-ring S . Therefore from Theorem 3.3 we obtain

THEOREM 4.1. *If a semisimple I -ring S is WR, then S has the following property: The right (left) annihilator of a left (right) ideal A is nonzero if and only if there exists a nonzero left (right) ideal B such that $A \cap B = 0$.*

Let S be a regular ring. If the lattice L of principal left ideals is complete, we say that S is a complete regular ring. A complete regular ring S is called continuous if L has the following property: $(\bigcup a_\alpha) \cap b = \bigcup (a_\alpha \cap b)$ and $(\bigcap a_\alpha) \cup b = \bigcap (a_\alpha \cup b)$ for any chain (a_α) and any element b in L . We proved in [4, Theorem 7] that a complete regular ring is continuous if and only if it satisfies (K_i) and (K_r) . Thus, by Theorem 3.3 we obtain

THEOREM 4.2. *A complete regular ring S is continuous if and only if $\bar{S}_l = \bar{S}_r$.*

By [3, Theorem 3] every continuous regular ring is (both left and right) self injective if it contains no nonzero strongly regular ideals. Hence we have

THEOREM 4.3. *Let S be a complete regular ring, and suppose that S does not contain any nonzero strongly regular ideals. Then $\bar{S}_l = \bar{S}_r$ (if and only if S is self injective, that is, $S = \bar{S}_l = \bar{S}_r$).*

REFERENCES

1. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. 2 (1951), 891–895.
2. Y. Utumi, *On quotient rings*, Osaka Math. J. 8 (1956), 1–18.
3. ———, *On continuous regular rings and semisimple self injective rings*, Canad. J. Math. 12 (1960), 597–605.
4. ———, *On continuous regular rings*, Canad. Math. Bull. 4 (1961), 63–69.

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INVERSE LIMITS OF SOLVABLE GROUPS

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In this paper we generalize to groups of Galois type some results of P. Hall on finite solvable groups [1; 2; 3]. We need, in a modified form, some results of van Dantzig: the definition of supernatural numbers (which are related to van Dantzig's universal numbers) and Theorem 5, which he proved for ordinary p -Sylow subgroups [6]. Lemmas 1 and 4 and the method of proof in Theorem 5 are due to Tate [5].

A topological group G is of *Galois type* if it is compact and totally disconnected. In any Galois type group the open normal subgroups form a neighborhood base at the identity. Every closed subgroup is the intersection of the open subgroups containing it [4]. Whenever M and N are open normal subgroups of G and $N \supset M$ we shall write ϕ_N^M for the natural homomorphism of G/M onto G/N (these quotient groups are finite) and ϕ_N for the natural homomorphism of G onto G/N . G is the inverse limit of the groups $\{G/N\}$, N ranging over the open normal subgroups of G . Conversely, the inverse limit of finite groups is of Galois type.

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