SIMPLE $(-1, 1)$ RINGS WITH AN IDEMPOTENT[1]

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1. **Introduction.** Algebras of type (γ, δ) were first defined by Albert [1]. They are algebras (nonassociative) satisfying the following identities:

$$A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0,$$

and

$$(z, x, y) + \gamma(x, z, y) + \delta(y, z, x) = 0,$$

where $\gamma^2 - \delta^2 + \delta = 1$, and where $(x, y, z) = (xy)z - x(yz)$.

Simple rings of type (γ, δ) have been studied by Kleinfeld and Kokoris. Their results show that except for types $(-1, 1)$ and $(1, 0)$, all simple (γ, δ) rings with an idempotent e which is not the unity element are associative [7]. In this paper this result is extended to types $(-1, 1)$ and $(1, 0)$.

2. **Basic identities.** When $\gamma = -1$ and $\delta = 1$ the defining identities of a (γ, δ) ring are equivalent to:

$$A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0, \tag{1}$$

and

$$B(x, y, z) = (x, y, z) + (x, z, y) = 0. \tag{2}$$

A ring of type $(1, 0)$ is anti-isomorphic to one of type $(-1, 1)$ [7, Theorem 1], therefore we will consider only rings of type $(-1, 1)$. The right alternative law, $(z, x, x) = 0$, is an immediate consequence of (1) and (2) since $0 = A(x, x, z) - B(x, x, z)$.

In any ring we have the identity $F(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0$. We also have the identity $(xy, z) - x(y, z) - (x, z)y - (x, y, z) + (x, z, y) - (z, x, y) = 0$ in any ring, where $(x, y) = xy - yx$. In a $(-1, 1)$ ring this becomes

$$C(x, y, z) = (xy, z) - x(y, z) - (x, z)y - 2(x, y, z) - (z, x, y) = 0,$$

because of (2).

The combination of (1) and $F(w, x, y, z) = 0$ as in [4] gives

$$G(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0.$$

Presented to the Society, January 29, 1960; received by the editors August 7, 1961.

[1] This paper is from the author's doctoral dissertation which was done under the guidance of Erwin Kleinfeld.

For all that follows we will assume that R is a ring of type $(-1, 1)$ of characteristic not 2 or 3 (i.e., no elements of R have additive order 2 or 3).

The next three identities are consequences of the right alternative law and are proved in [3].

$$H(x, y, z) = (x, y, yz) - (x, y, z)y = 0,$$

$$J(x, w, y, z) = (x, w, yz) + (x, y, wz) - (x, w, z)y - (x, y, z)w = 0,$$

$$K(x, y, z) = (x, y^2, z) - (x, y, yz + zy) = 0.$$

From $G(x, x, x, y) + (x, B(x, y, x)) = 0$ it follows that $2(x, (x, x, y)) = 0$. From this and the fact that $(x, y, x) = -(x, x, y)$ we obtain

$$(x, (x, x, y)) = 0 \quad \text{and} \quad (x, (x, y, x)) = 0. \tag{3}$$

Now $G(x, y, x, y) = 0$ gives $2(x, (y, x, y)) - 2(y, (x, y, x)) = 0$ and thus $(x, (y, x, y)) - (y, (x, y, x)) = 0$. From $B(y, y, x) = 0$ and $B(x, x, y) = 0$ we then have $(x, (y, y, x)) - (y, (x, x, y)) = 0$. Combining this with $G(x, y, y, x) = 0$ gives $2(x, (y, y, x)) = 0$ and therefore

$$(x, (y, y, x)) = 0. \tag{4}$$

Replacing x with $x + z$ in (4) gives

$$L(x, y, z) = (x, (y, y, z)) + (z, (y, y, x)) = 0.$$

Replacing y with $y + z$ in (4) gives

$$M(x, y, z) = (x, (y, z, x)) + (x, (z, y, x)) = 0.$$

From $0 = F(x, x, x, y) - H(x, x, y) + (x, (x, x, y))$ we see that $(x^2, x, y) = (x, x^2, y)$ and thus from $M(y, x^2, x) = 0$ we have $2(y, (x^2, x, y)) = 0$. This gives

$$(y, (x^2, x, y)) = 0. \tag{5}$$

We now use $B(y, x^2, x) - H(y, x, x) = 0$ to get $(y, x^2, x) = 0$. Combining this with $G(y, y, x^2, x) + L(x, y, x^2) = 0$ and (5) gives

$$(x^2, (y, y, x)) = 0. \tag{6}$$

The identity $(xy, z) + (yz, x) + (zx, y) = A(x, y, z)$ holds in any ring hence in a $(-1, 1)$ ring where $A(x, y, z) = 0$ we have

$$D(x, y, z) = (xy, z) + (yz, x) + (zx, y) = 0.$$

Since x and x^2 commute with (y, y, x), we have $D(x, x, (y, y, x)) = 2((y, y, x)x, x) = 0$ and thus $((y, y, x)x, x) = 0$. Now we use $(y, y, x)x = (y, y, x)x - B(y, y, x)x - H(y, x, y) = -(y, x, xy)$ and we get $(x, (y, x, xy)) = 0$. Replacing y with $y + w$ gives

$$(x, (y, x, xw)) + (x, (w, x, xy)) = 0. \tag{7}$$

From $L(x, y, x^2) = 0$ and (6) we get $(x, (y, y, x^2)) = 0$. Now since $0 = (x, B(y, y, x^2)) - (x, (y, y, x^2)) - (x, K(y, x, y)) = (x, (y, x, xy)) + (x, (y, x, yx))$ and $(x, (y, x, xy)) = 0$ we have $(x, (y, x, yx)) = 0$. Clearly $(x, (y, yx, x)) = 0$, since $B(y, x, yx) = 0$ and thus $0 = (x, (y, yx, x)) - M(x, y, yx) + (x, B(yx, y, x)) = (x, (yx, x, y))$. Therefore $(x, A(yx, x, y)) = 0$ gives $(x, (x, y, yx)) = 0$. Replacing y with $y+w$ in $(x, (x, y, yx)) = 0$ and in $(x, (yx, x, y)) = 0$ gives respectively:

$$\begin{aligned} &(x, (x, w, yx)) + (x, (x, y, wx)) = 0, \\ &(x, (wx, x, y)) + (x, (yx, x, w)) = 0. \end{aligned} \tag{8}$$

We now use $F(w, x, x, y) = 0$ to get $w(x, x, y) = (wx, x, y) - (w, x^2, y) + (w, x, xy) + K(w, x, y) = (wx, x, y) - (w, x, yx)$. Commuting this with x gives $(x, w(x, x, y)) = (x, (wx, x, y)) - (x, (w, x, yx)) = (x, (wx, x, y)) - (x, (w, x, yx)) + (x, B(w, x, yx)) - M(x, w, yx) + (x, B(yx, x, w)) = (x, (wx, x, y)) + (x, (yx, x, w))$ and this is zero by (8). Therefore

$$(x, w(x, x, y)) = 0. \tag{9}$$

The next step is to use $0 = F(x, y, x, w) + F(x, y, w, x) - B(xy, x, w) + xB(y, x, w) + B(x, yx, w)$ to get $(x, y, x)w = (x, w, yx) + (x, y, wx) - (x, yw, x) + (x, y, xw) - (x, y, w)x$. From (8) we see that x commutes with $(x, w, yx) + (x, y, wx)$ and from (3) we see that x commutes with (x, yw, x). We wish now to show that x commutes with $(x, y, xw) - (x, y, w)x$. To do this we observe from $0 = -A(x, y, w)x + B(y, w, x)x + H(y, x, w) - H(w, x, y) - B(y, x, xw) + A(x, y, xw)$ that $(x, y, xw) - (x, y, w)x = (w, x, xy) - (xw, x, y)$. Thus $(x, (x, y, xw)) - (x, (x, y, w)x) = (x, (w, x, xy)) - (x, (xw, x, y)) + (x, B(xw, x, y)) - M(x, xw, y) + (x, B(y, xw, x)) = (x, (w, x, xy)) + (x, (y, x, xw))$ and this is zero by (7). Thus $(x, (x, y, x)w) = 0$ and since $(x, x, y) = -(x, y, x)$ we have

$$(x, (x, x, y)w) = 0. \tag{10}$$

An immediate consequence of (9), (10), (3) and $D((x, x, y), w, x) - D(w, (x, x, y), x) = 0$ is $((w, x), (x, x, y)) = 0$. Then from $L(y, x, (w, x)) = 0$ we get

$$(y, (x, x, (x, w))) = 0. \tag{11}$$

We conclude this section with two remarks. The first is that

$$(x, x, y)^2 = 0. \tag{12}$$

To prove this let $S(a, b) = \{x \in R \mid (x, a, b) = x(b, a)\}$. Then (x, x, y)

and $(x, x, y)x$ are elements of $S(x, y)$ by Lemma 4 of [3]. In R, $(x, x, y)x = x(x, x, y)$, thus by Lemma 3 of [3] we get (12).

For the second remark we define U to be the set of elements u of R which commute with all elements of R. Let $u \in U$. Then $C(x, x, u) = 0$ gives $-2(x, x, u) = 0$. Hence $(x, x, u) = 0$, and $(x, u, x) = 0$ because of (2). Replacing x by $x+y$ in these last two identities gives

$$(13) \qquad \begin{aligned} (x, y, u) &= -(y, x, u) \quad \text{and} \\ (x, u, y) &= -(y, u, x) \quad \text{for all } u \in U. \end{aligned}$$

3. **Identities involving an idempotent.** Let e be an element of R such that $e^2 = e \neq 0$. From $(x, K(e, e, y)) = 0$ and (11) we obtain $(x, (e, e, y)) - 2(x, (e, e, ey)) = 0$. From $C(e, e, (e, e, y)) = 0$ we get $-2(e, e, (e, e, y)) = 0$ because of (3), and thus $(e, e, (e, e, y)) = 0$. Therefore we can replace y by ey in $(x, (e, e, y)) - 2(x, (e, e, ey)) = 0$ to get $-(x, (e, e, ey)) = 0$ and finally

$$(14) \qquad (x, (e, e, y)) = 0,$$

i.e., an associator of the form (e, e, y) commutes with all elements of R.

We have from (12) that $(e, e, y)^2 = 0$. Replacing y by $y+w$ and using (14) gives $2(e, e, y)(e, e, w) = 0$ and thus

$$(15) \qquad (e, e, y)(e, e, w) = 0.$$

Let u be substituted for (e, e, y). From $C(e, e, xu) = 0$ we have $-2(e, e, xu) = 0$ since $(e, xu) = 0$ by (9). Hence $(e, e, xu) = 0$ and $F(e, e, x, u) = 0$ gives

$$(16) \qquad (e, x, u) = (e, ex, u) + e(e, x, u),$$

because of (15).

We will transform (16) in three successive stages. First we observe that $G(e, e, x, y) = 0$ and $(y, (e, e, x)) = 0$ give $(e, (e, x, y)) - (e, (x, y, e)) = 0$. From this and $0 = (e, A(e, x, y)) - (e, B(y, e, x)) + M(e, y, x)$ we have $3(e, (x, y, e)) = 0$. Hence $(e, (x, y, e)) = 0$ and $(e, (e, x, y)) = 0$ for all x, y in R. This reduces (16) to

$$(17) \qquad (e, x, u) = (e, ex, u) + (e, x, u)e.$$

Now since $u \in U$ we conclude from (13) and from $0 = A(e, x, u) + B(e, x, u)$ that $2(e, x, u) + (u, e, x) = 0$. Similarly $2(e, ex, u) + (u, e, ex) = 0$. Multiplying $2(e, x, u) + (u, e, x) = 0$ on the right by e and subtracting from $2(e, ex, u) + (u, e, ex) = 0$ gives $2(e, ex, u) - 2(e, x, u)e = 0$ since $H(u, e, x) = 0$. This gives $(e, ex, u) - (e, x, u)e = 0$ and (17) becomes

$$(18) \qquad (e, x, u) = 2(e, ex, u).$$

The next step follows from the easy observation that if $u_1, u_2 \in U$ then $(x, u_1, u_2) = (u_1, x, u_2) = (u_1, u_2, x) = 0$ for all x in R. This follows from $A(x, u_1, u_2) = 0$, $B(x, u_1, u_2) = 0$, and (13). Thus since (e, e, x) and u are in U we have $(e, (e, e, x), u) = 0$ and we can replace x by ex in (18) to get $(e, ex, u) = 2(e, ex, u)$. Hence $(e, ex, u) = 0$ and

$$(e, x, u) = 0. \tag{19}$$

From (19), (13), (1) and (2) it is easily seen that

$$(x, u, e) = (u, e, x) = 0. \tag{20}$$

Since $ue = (e, e, y)e + H(e, e, y) = (e, e, ey)$, we have that $0 = C(u, e, x) = -u(e, x)$ by (14) and (20). Thus using (14) again we have

$$(e, x)(e, e, y) = 0. \tag{21}$$

We are now finally able to use $0 = J(e, y, e, (e, x)) - A(e, y, e(e, x)) + A(e, y, (e, x))e + B(y, e(e, x), e) - B(y, (e, x), e)e - H(y, e, (e, x)) + H((e, x), e, y) - (C(e, e, x), e, y) - F((e, x), e, e, y)$, along with the right alternative law, (20), and (21) to get

$$(e, e, (e, x))y = (e, e, y(e, x)). \tag{22}$$

LEMMA 1. *Let R be a commutative $(-1, 1)$ ring of characteristic not 3. Then R is associative.*

PROOF. For a commutative $(-1, 1)$ ring $C(y, x, x) = 0$ gives $(x, y, x) = 0$ and R can easily be seen to be alternative. In an alternative ring $A(x, y, z) = 3(x, y, z)$. Therefore $(x, y, z) = 0$.

LEMMA 2. *Let R be a simple, not associative, ring of type $(-1, 1)$ of characteristic not 2 or 3. Let $S = \{s \in R \mid s \in U$ and $sy \in U$ for all $y \in R\}$, where $U = \{u \in R \mid (u, y) = 0$ for all $y \in R\}$. Then $S = 0$.*

PROOF. Let $s \in S$ and $x, y \in R$. Then using the fact that s and sx are in U and using (13) we have the following: $(sx)y = (s, x, y) + s(xy)$ and $(sx)y = y(xs) = -(y, x, s) + (yx)s = (x, y, s) + s(yx)$. Combining these, we have $3(sx)y = (s, x, y) + 2(x, y, s) + s(xy + 2yx)$. Since $s \in U$, we have $(x, y, s) = (y, s, x)$ because of (13) and (2), and $0 = A(s, x, y) = (s, x, y) + 2(x, y, s)$. From this we have $3(sx)y \in U$. Now the set $T = \{3x \mid x \in R\}$ is an ideal of R and is not zero, thus $T = R$. This means that S is an ideal of R because $3y$ is an arbitrary element of R and $(sx)(3y) \in U$. If $S = R$, R is commutative hence associative by Lemma 1. Thus $S = 0$.

LEMMA 3. *Let R be a simple not associative ring of type $(-1, 1)$ of characteristic not 2 or 3. Let e be an idempotent of R. Then we have the following*:

$$(e, e, x) = (e, x, e) = 0. \tag{23}$$

REMARK. This is false if R is not simple as will be shown by an example.

PROOF. From (14) we have $(e, e, y) \in U$ for all y in R. Thus $(e, e, (e, x)) \in U$. From (22) we have $(e, e, (e, x))y = (e, e, y(e, x))$ and this again is in U. Thus $(e, e, (e, x)) \in S$. But $S=0$ by Lemma 2 and we have $(e, e, (e, x))=0$. This applied to $0=K(e, e, x)$ gives $(e, e, x) - 2(e, e, ex)=0$. Observing again that $(e, e, (e, e, x))=0$ we can replace x by ex in $(e, e, x)-2(e, e, ex)=0$ to get $-(e, e, ex)=0$ and therefore $(e, e, x)=0$. From $B(e, e, x)=0$ we get the rest of (23).

4. **Main section.** For the following remarks and the lemmas which follow, R will be assumed to be a simple $(-1, 1)$ ring which is not associative and which is of characteristic not 2 or 3. For Lemmas 4 and 5 R is also assumed to have an idempotent $e \neq 0$.

Before proceeding to Lemma 4 we first observe that, when R has an idempotent e, we have $R = R_{11} + R_{10} + R_{01} + R_{00}$ where $R_{ij} = \{x \in R \mid ex = ix \text{ and } xe = jx\}$. This follows from (23) and the fact that $x = exe + (ex - exe) + (xe - exe) + (x - ex - xe + exe)$, and the right alternative law.

LEMMA 4. *The multiplication for the spaces* R_{ij} *is as follows*:

(1) $R_{ij}R_{jk} \subset R_{ik}$;

(2) $R_{ij}R_{km} = 0$ *for* $j \neq k$ *except when* $i = k$ *and* $j = m$;

(3) $R_{10}R_{10} \subset R_{11}$ *and* $R_{01}R_{01} \subset R_{00}$. (*This differs from the associative situation only in* (3)).

PROOF. When $x \in R_{ij}$, $y \in R_{ji}$ we have directly that $(x, e, y) = (y, e, x) = 0$. From $B(x, e, y)=0$ and $B(y, e, x)=0$ we have $(x, y, e) = (y, x, e) = 0$. Thus from $A(x, y, e)=0$ and $A(y, x, e)=0$ we get $(e, x, y) = (e, y, x) = 0$. Thus we can easily see that $R_{ij}R_{ji} \subset R_{ii}$.

Next we take $x \in R_{ij}$, $y \in R_{ii}$ where $i \neq j$. We can compute directly that $(y, e, x)=0$. From $K(x, e, y) = (-1)^i(x, e, y) = 0$ we get $(x, e, y) = 0$. It follows from $B(x, e, y)=0$, $B(y, e, x)=0$, $A(x, y, e)=0$, and $A(y, x, e)=0$ that $(x, y, e) = (y, x, e) = (e, x, y) = (e, y, x) = 0$. Thus $R_{ij}R_{ii}=0$ and $R_{ii}R_{ij} \subset R_{ij}$.

When $x \in R_{00}$ and $y \in R_{11}$ we have $K(x, e, y) = -(x, e, y) = 0$ and thus $xy=0$. We also have $K(y, e, x) = (y, e, x) = 0$ and thus $yx=0$. This gives $R_{00}R_{11} = R_{11}R_{00} = 0$.

When $x \in R_{00}$ and $y \in R_{10}$ we have $(y, e, x)=0$ by direct computation. Thus $A(e, x, y) - B(x, y, e) = 0$ gives $(e, x, y) - (x, e, y) = 0$. Hence $e(xy) = xy$. From $B(y, e, x)=0$ and $(y, e, x)=0$ we get $(y, x, e)=0$ and so $A(e, y, x)=0$ gives $(e, y, x) + (x, e, y) = 0$. This can be expanded to give $yx - e(yx) - xy = 0$. Multiplying this last equation on the left by e gives $-e(xy)=0$ in view of (23). Hence $xy=0$. Thus $e(yx) = yx$.

From $(y, x, e)=0$ we have $(yx)e=0$. Therefore $R_{00}R_{10}=0$ and $R_{10}R_{00} \subset R_{10}$.

When $x\in R_{01}$ and $y\in R_{11}$ we have $(x, e, y)=0$ by direct calculation. Thus $B(x, y, e)=0$ gives $(x, y, e)=0$. Thus $(xy)e=xy$. Now $0=H(y, e, x)$ gives $(y, e, x)e=0$. Therefore $A(e, x, y)=0$ along with $(x, y, e)=0$ gives $(e, x, y)e=0$. By the remark just before (17) we have $(e, (e, z, w))=0$ for all z, w in R. Hence $e(e, x, y)=0$ and $e(e(xy))=0$ and $e(xy)=0$ by (23). Therefore $xy\in R_{01}$. Since $e(xy)=0$, $(e, x, y)=0$. Combining this with $A(e, x, y)=0$ and $(x, y, e)=0$ gives $(y, e, x)=0$ and $yx=0$. Thus $R_{01}R_{11}\subset R_{01}$ and $R_{11}R_{01}=0$.

For the final case $x, y\in R_{ij}$ where $i\neq j$ we have $H(x, e, y)=0$ combined with $(e, (x, e, y))=0$ to give $(x, e, y)\in R_{ii}$. But $(x, e, y)=\pm xy$. Hence $xy\in R_{ii}$.

LEMMA 5. *For all $x, y\in R$ and the idempotent e the following identity holds*:

$$(e, x, y) = 0. \tag{24}$$

PROOF. For $x, y\in R$ we have that $x=x_{11}+x_{10}+x_{01}+x_{00}$ and $y=y_{11}+y_{10}+y_{01}+y_{00}$ where $x_{ij}, y_{ij}\in R_{ij}$. Direct calculation and Lemma 4 yield (24).

LEMMA 6. *Every element of R is the sum of a finite number of associators.*

PROOF. Let S be the class of elements in R which are sums of a finite number of associators. Since R is not associative $S\neq 0$. We will show that S is an ideal of R. From $F(w, x, y, z)+F(w, y, x, z)+B(w, x, y)z=0$ we have $w((x, y, z)+(y, x, z))\in S$ for all $w, x, y, z\in R$. Therefore $wA(x, y, z)+wB(x, y, z)+w((x, y, z)+(y, x, z))-w((x, z, y)+(z, x, y))-wB(y, x, z)=3w(x, y, z)$ is an element of S. However $3w$ is an arbitrary element of R, and so $RS\subset S$. Next from $F(w, x, y, z)=0$ we see that $SR\subset RS+S\subset S$ and S is an ideal not zero. Hence $S=R$.

The center of R is defined to be the elements $n\in U$ with the property that $(n, x, y)=(x, n, y)=(x, y, n)=0$ for all $x, y\in R$.

LEMMA 7. *Let $n\in R$ satisfy $(n, x, y)=0$ for all $x, y\in R$. Then n is in the center of R.*

PROOF. We must first show that $n\in U$. To do this we need only show that n commutes with any associator because of Lemma 6. $M(x, n, y)=0$ gives $(x, (y, n, x))=0$ since $(n, y, x)=0$. Thus from $(x, A(n, x, y))=0$ we get $(x, (x, y, n))=0$. Combining these results with $G(x, y, n, x)=0$ gives $(n, (x, x, y))=0$. Replacing x with $x+z$

in this last identity gives $(n, (x, z, y))+(n, (z, x, y))=0$ for all x, y, z in R. Thus $0 = (n, A(x, y, z)) + ((n, (x, y, z)) + (n, (y, x, z))) - ((n, (x, z, y))+(n, (z, x, y)))+(n, B(x, y, z))-(n, B(y, z, x)) = 3(n, (x, y, z))$ and $(n, (x, y, z))=0$ for any associator (x, y, z) in R. We now have $n \in U$ because every element of R is the sum of associators. Because of (13) we also have $A(n, x, y)+B(x, y, n)=2(x, y, n) =0$. Therefore $(x, y, n)=0$ and from $A(n, x, y)=0$ we have $(y, n, x) =0$. This shows that n is in the center of R.

THEOREM. *Let R be a simple ring of type* $(-1, 1)$. *Let R have characteristic not* 2 *or* 3. *Let e be an idempotent of R such that* $e \neq 0, 1$. *Then R is associative.*

PROOF. Assume R is not associative. Then Lemma 5 gives $(e, x, y) =0$ for all x, y in R. Lemma 7 then gives us that e is in the center of R. Therefore $R_{01}=R_{10}=0$ and $R=R_{11}+R_{00}$. But then R_{00} is an ideal of R by Lemma 4 and $e \notin R_{00}$. Thus $R_{00}=0$ and $R=R_{11}$, a contradiction since e is the unity element of R_{11}.

5. Example. The following example is given by Albert in [2] as a right alternative algebra. It can be shown to satisfy (1) and (2) hence it is a $(-1, 1)$ algebra. It is finite dimensional and Lemma 3 fails for it.

Let F be a field and A an algebra with basis elements e, u, v, w, z. We define multiplication of basis elements as follows: $e^2=e$, $eu=v$, $ev=0$, $ue=u$, $ew=w-z$, $ez=ze=z$ and all other products are zero.

BIBLIOGRAPHY

1. A. A. Albert, *Almost alternative algebras*, Portugal. Math. **8** (1949), 23–36.
2. ———, *On the right alternative algebras*, Ann. of Math. (2) **50** (1949), 318–328.
3. E. Kleinfeld, *Right alternative rings*, Proc. Amer. Math. Soc. **4** (1953), 939–944.
4. ———, *Rings of* (γ, δ) *type*, Portugal. Math. **18** (1959), 107–110.
5. ———, *Simple algebras of type* (1, 1) *are associative*, Canad. J. Math. **13** (1961), 129–148.
6. L. A. Kokoris, *On a class of almost alternative algebras*, Canad. J. Math. **8** (1956), 250–255.
7. ———, *On rings of* (γ, δ) *type*, Proc. Amer. Math. Soc. **9** (1958), 897–904.

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