1. Introduction. Algebras of type \((\gamma, \delta)\) were first defined by Albert [1]. They are algebras (nonassociative) satisfying the following identities:

\[ A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0, \]

and

\[ (z, x, y) + \gamma(x, z, y) + \delta(y, z, x) = 0, \]

where \(\gamma^2 - \delta^2 + \delta = 1\), and where \((x, y, z) = (xy)z - x(yz)\).

Simple rings of type \((\gamma, \delta)\) have been studied by Kleinfeld and Kokoris. Their results show that except for types \((-1, 1)\) and \((1, 0)\), all simple \((\gamma, \delta)\) rings with an idempotent \(e\) which is not the unity element are associative [7]. In this paper this result is extended to types \((-1, 1)\) and \((1, 0)\).

2. Basic identities. When \(\gamma = -1\) and \(\delta = 1\) the defining identities of a \((\gamma, \delta)\) ring are equivalent to:

\[ A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0, \]

and

\[ B(x, y, z) = (x, y, z) + (y, z, x) = 0. \]

A ring of type \((1, 0)\) is anti-isomorphic to one of type \((-1, 1)\) [7, Theorem 1], therefore we will consider only rings of type \((-1, 1)\). The right alternative law, \((z, x, x) = 0\), is an immediate consequence of (1) and (2) since \(0 = A(x, x, z) - B(x, x, z)\).

In any ring we have the identity \(F(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) = 0\). We also have the identity \((xy, z) - (x, y, z) - (x, z)y - (x, y, z) + (x, z, y) - (z, x, y) = 0\) in any ring, where \((x, y) = xy - yx\). In a \((-1, 1)\) ring this becomes

\[ C(x, y, z) = (xy, z) - x(y, z) - (x, z)y - 2(x, y, z) - (z, x, y) = 0, \]

because of (2).

The combination of (1) and \(F(w, x, y, z) = 0\) as in [4] gives

\[ G(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0. \]
For all that follows we will assume that $R$ is a ring of type $(-1, 1)$ of characteristic not 2 or 3 (i.e., no elements of $R$ have additive order 2 or 3).

The next three identities are consequences of the right alternative law and are proved in [3].

$$H(x, y, z) = (x, y, yz) - (x, y, z)y = 0,$$

$$J(x, w, y, z) = (x, w, yz) + (x, y, wz) - (x, w, z)y - (x, y, z)w = 0,$$

$$K(x, y, z) = (x, y^2, z) - (x, y, yz + yz) = 0.$$

From $G(x, x, x, y) + (x, B(x, y, x)) = 0$ it follows that $2(x, (x, x, y)) = 0$. From this and the fact that $(x, y, x) = -(x, x, y)$ we obtain

$$(x, (x, x, y)) = 0 \text{ and } (x, (x, y, x)) = 0.$$

Now $G(x, y, x, y) = 0$ gives $2(x, (y, x, y)) - 2(y, (x, x, y)) = 0$ and thus $(x, (y, x, y)) - (y, (x, x, y)) = 0$. From $B(y, y, x) = 0$ and $B(x, x, y) = 0$ we then have $(x, (y, x, y)) - (y, (x, x, y)) = 0$. Combining this with $G(x, y, y, x) = 0$ gives $2(x, (y, y, x)) = 0$ and therefore

$$(x, (y, y, x)) = 0.$$

Replacing $x$ with $x + z$ in (4) gives

$$L(x, y, z) = (x, (x, x, y)) + (z, (y, y, x)) = 0.$$

Replacing $y$ with $y + z$ in (4) gives

$$M(x, y, z) = (x, (y, z, x)) + (x, (y, x, z)) = 0.$$

From $0 = F(x, x, y, y) - H(x, x, y) + (x, (x, x, y))$ we see that $(x^2, x, y) = (x, x^2, y)$ and thus from $M(y, x^2, x) = 0$ we have $2(y, (x^2, x, y)) = 0$. This gives

$$(y, (x^2, x, y)) = 0.$$

We now use $B(y, x^2, x) - H(y, x, x) = 0$ to get $(y, x^2, x) = 0$. Combining this with $G(y, y, x^2, x) + L(x, y, x^2) = 0$ and (5) gives

$$(x^2, (y, y, x)) = 0.$$

The identity $(xy, z) + (yz, x) + (zx, y) = A(x, y, z)$ holds in any ring hence in a $(-1, 1)$ ring where $A(x, y, z) = 0$ we have

$$D(x, y, z) = (xy, z) + (yz, x) + (zx, y) = 0.$$
From $L(x, y, x^2) = 0$ and (6) we get $(x, (y, y, x^2)) = 0$. Now since $0 = (x, B(y, y, x^2)) - (x, (y, y, x^2)) - (x, K(y, x, y)) = (x, (y, x, xy)) + (x, (y, x, xy))$ and $(x, (y, x, xy)) = 0$ we have $(x, (y, x, yx)) = 0$. Clearly $(x, (y, yx, x)) = 0$, since $B(y, y, yx) = 0$ and thus $0 = (x, (y, yx, x)) - M(x, y, yx) + (x, B(yx, x, y)) = (x, (yx, x, y))$. Therefore $(x, A(yx, x, y)) = 0$ gives $(x, (x, y, yx)) = 0$. Replacing $y$ with $y + w$ in $(x, (x, y, yx)) = 0$ and in $(x, (yx, x, y)) = 0$ gives respectively:

$$(x, (x, w, yx)) + (x, (x, y, wx)) = 0,$$

$$B(x, w, x, y) + (x, B(yx, x, y)) = (x, (yx, x, y)).$$

We now use $F(w, x, y) = 0$ to get $w(x, x, y) = (wx, x, y) - (w, x^2, y) + (w, x, xy) + K(w, x, y) = (wx, x, y) - (w, x, yx)$. Commuting this with $x$ gives $(x, w(x, x, y)) = (x, (wx, x, y)) - (x, (w, y, yx)) = (x, (wx, x, y)) - (x, (w, x, yx)) + (x, B(x, w, y)) - M(x, w, yx) = (x, B(yx, x, w)) = (x, (wx, x, y)) + (x, (yx, x, w))$ and this is zero by (8). Therefore

$$(x, w(x, x, y)) = 0.$$

The next step is to use $0 = F(x, y, x, w) + F(x, y, w, x) - B(xy, x, w) + xB(y, x, w) + B(x, yx, w)$ to get $(x, y, x)w = (x, w, yx) + (x, y, wx) - (x, yw, x) + (x, y, yx)$ from (8) we see that $x$ commutes with $(x, w, yx) + (x, y, wx)$ and from (3) we see that $x$ commutes with $(x, yw, x)$. We wish now to show that $x$ commutes with $(x, y, xw)$ and $(x, y, w)x$. To do this we observe from $0 = -A(x, y, w)x + B(y, y, w)x + H(y, y, w) - H(y, w, y) - B(y, x, xw) + A(x, y, xw)$ that $(x, y, xw)$ and $(x, y, w)x$ commute with $(x, yw, x)$. Thus $(x, (x, y, xw)) - (x, (x, y, w)x) = (x, (w, x, y)) - (x, (xw, x, y)) + (x, B(xw, y, x)) - M(x, xw, y) + (x, B(y, yw, x)) = (x, (wx, x, y)) + (x, (yx, x, w))$ and this is zero by (7). Thus $(x, (x, y, xw)) = 0$ and since $(x, x, y) = - (x, y, x)$ we have

$$(x, (x, x, y)w) = 0.$$

An immediate consequence of (9), (10), (3) and $D((x, x, y), w, x) - D(w, (x, x, y), x) = 0$ is $(w, x, (x, x, y)) = 0$. Then from $L(y, x, (w, x)) = 0$ we get

$$(y, (x, x, (x, w))) = 0.$$

We conclude this section with two remarks. The first is that

$$(x, x, y)^2 = 0.$$

To prove this let $S(a, b) = \{ xR | (x, a, b) = (x, b, a) \}$. Then $(x, x, y)$ and $xR = xR$. The second remark is that
and \((x, x, y)x\) are elements of \(S(x, y)\) by Lemma 4 of [3]. In 
\(R, (x, x, y)x = x(x, x, y)\), thus by Lemma 3 of [3] we get (12).

For the second remark we define \(U\) to be the set of elements \(u\) of 
\(R\) which commute with all elements of \(R\). Let \(u \in U\). Then 
\(C(x, x, u) = 0\) gives \(-2(x, x, u) = 0\). Hence \((x, x, u) = 0\), and 
\((x, u, x) = 0\) because of (2). Replacing \(x\) by \(x + y\) in these last two identities gives 
\(\begin{align*}
(x, y, u) &= -(y, x, u) \\
(x, u, y) &= -(y, u, x)
\end{align*}\) for all \(u \in U\).

3. **Identities involving an idempotent.** Let \(e\) be an element of \(R\) 
such that \(e^2 = e\neq 0\). From (12) and (11) we obtain 
\(\begin{align*}
(x, e, e, y) - 2(x, e, e, ey) &= 0. \\
C(e, e, e, (e, e, y)) &= 0.
\end{align*}\) From \(C(e, e, e, y) = 0\) we get 
\(-2(e, e, e, y) = 0\) because of (3), and thus \((e, e, e, y) = 0\). Therefore we can replace \(y\) by \(ey\) in 
\(\begin{align*}
(x, e, e, y) - 2(x, e, e, ey) &= 0.
\end{align*}\) and finally 
\(\begin{align*}
(x, (e, e, y)) &= 0, \\
(x, e, e, y) &= 0,
\end{align*}\) i.e., an associator of the form \((e, e, y)\) commutes with all elements of \(R\).

We have from (12) that \((e, e, y)^2 = 0\). Replacing \(y\) by \(y + w\) and using 
(14) gives \(2(e, e, y)(e, e, w) = 0\) and thus 
\(\begin{align*}
(e, e, y)(e, e, w) &= 0.
\end{align*}\)

Let \(u\) be substituted for \((e, e, y)\). From \(C(e, e, xu) = 0\) we have 
\(\begin{align*}
-2(e, e, xu) &= 0 \\
F(e, e, x, u) &= 0.
\end{align*}\) From this and \(0 = (e, A(e, x, y)) - (e, B(y, e, x)) = 0\) we have 
\(\begin{align*}
3(e, (x, y, e)) &= 0. \\
G(e, (x, y, e)) &= 0 \text{ for all } x, y \in R.
\end{align*}\) This reduces (16) to 
\(\begin{align*}
(e, x, u) &= (e, xu) + e(x, u),
\end{align*}\) because of (15).

We will transform (16) in three successive stages. First we observe that 
\(\begin{align*}
G(e, e, x, y) &= 0 \text{ and } (y, (e, e, x)) = 0 \text{ give } (e, (e, x, y)) - (e, (x, y, e)) &= 0. \\
F(e, e, x, u) &= 0.
\end{align*}\) From this and \(0 = (e, A(e, x, y)) - (e, B(y, e, x)) = 0\) we have 
\(\begin{align*}
3(e, (x, y, e)) &= 0. \\
G(e, (x, y, e)) &= 0 \text{ for all } x, y \in R.
\end{align*}\) This reduces (16) to 
\(\begin{align*}
(e, x, u) &= (e, xu) + (e, x, u).
\end{align*}\)

Now since \(u \in U\) we conclude from (13) and from 
\(0 = A(e, x, u) + B(e, x, u)\) that \(2(e, x, u) + (u, e, x) = 0\). Similarly \(2(e, e, xu) + (u, e, ex) = 0.\) Multiplying \(2(e, x, u) + (u, e, x) = 0\) on the right by \(e\) and subtracting from 
\(2(e, ex, u) + (u, e, ex) = 0\) gives \(2(e, ex, u) - 2(e, x, u)e = 0\) since \(H(u, e, x) = 0\). This gives 
\(e, ex, u) - (e, x, u)e = 0\) and (17) becomes 
\(\begin{align*}
(e, x, u) &= 2(e, ex, u).
\end{align*}\)
The next step follows from the easy observation that if \( u_1, u_2 \in U \) then \( (x, u_1, u_2) = (u_1, x, u_2) = (u_2, x, u_1) = 0 \) for all \( x \in R \). This follows from \( A(x, u_1, u_2) = 0 \), \( B(x, u_1, u_2) = 0 \), and (13). Thus since \( (e, e, x) \) and \( u \) are in \( U \) we have \( (e, (e, e, x), u) = 0 \) and we can replace \( x \) by \( ex \) in (18) to get \( (e, ex, u) = 2(e, ex, u) \). Hence \( (e, ex, u) = 0 \) and

\[
(19) \quad (e, x, u) = 0.
\]

From (19), (13), (1) and (2) it is easily seen that

\[
(20) \quad (x, u, e) = (u, e, x) = 0.
\]

Since \( ue = (e, e, y)e + H(e, e, y) = (e, e, ey) \), we have that \( 0 = C(u, e, x) = -u(e, x) \) by (14) and (20). Thus using (14) again we have

\[
(21) \quad (e, ex, e) = 0.
\]

We are now finally able to use

\[
0 = J(e, y, x, (e, x)) - A(e, y, e(e, x)) + A(e, y, (e, x)e + B(y, e(e, x), e) - B(y, e(e, x), e) - H(y, e, (e, x))
\]

along with the right alternative law, (20), and (21) to get

\[
(22) \quad (e, e, (e, x)y = (e, e, y(e, x)).
\]

**Lemma 1.** Let \( R \) be a commutative \((-1, 1)\) ring of characteristic not 3. Then \( R \) is associative.

**Proof.** For a commutative \((-1, 1)\) ring \( C(y, x, x) = 0 \) gives \( (x, y, x) = 0 \) and \( R \) can easily be seen to be alternative. In an alternative ring \( A(x, y, z) = 3(x, y, z) \). Therefore \( (x, y, z) = 0 \).

**Lemma 2.** Let \( R \) be a simple, not associative, ring of type \((-1, 1)\) of characteristic not 2 or 3. Let \( S = \{s \in R | s \in U \text{ and } sy \in U \text{ for all } y \in R\} \), where \( U = \{u \in R | (u, y) = 0 \text{ for all } y \in R\} \). Then \( S = 0 \).

**Proof.** Let \( s \in S \) and \( x, y \in R \). Then using the fact that \( s \) and \( sx \) are in \( U \) and using (13) we have the following: \((sx)y = (s, x, y) + s(xy) \) and \((sx)y = y(xs) = -(y, x, s) + (yx)s = (x, y, s) + s(yx) \). Combining these, we have \( 3(sx)y = (s, x, y) + 2(x, y, s) + s(xy + 2yx) \). Since \( s \in U \), we have \( (x, y, s) = (y, s, x) \) because of (13) and (2), and \( 0 = A(s, x, y) = (s, x, y) + 2(x, y, s) \). From this we have \( 3(sx)y \in U \). Now the set \( T = \{3x | x \in R\} \) is an ideal of \( R \) and is not zero, thus \( T = R \). This means that \( S \) is an ideal of \( R \) because \( 3y \) is an arbitrary element of \( R \) and \( (sx)(3y) \in U \). If \( S = R \), \( R \) is commutative hence associative by Lemma 1. Thus \( S = 0 \).

**Lemma 3.** Let \( R \) be a simple not associative ring of type \((-1, 1)\) of characteristic not 2 or 3. Let \( e \) be an idempotent of \( R \). Then we have the following:

\[
(23) \quad (e, e, x) = (e, x, e) = 0.
\]
Remark. This is false if \( R \) is not simple as will be shown by an example.

Proof. From (14) we have \((e, e, y) \in U\) for all \( y \) in \( R \). Thus \((e, e, (e, x)) \in U\). From (22) we have \((e, e, (e, x))y = (e, e, y(e, x))\) and this again is in \( U \). Thus \((e, e, (e, x)) \in S\). But \( S = 0 \) by Lemma 2 and we have \((e, e, (e, x)) = 0\). This applied to \( 0 = K(e, e, x) \) gives \((e, e, x) - 2(e, e, ex) = 0\). Observing again that \((e, e, (e, e, x)) = 0\) we can replace \( x \) by \( ex \) in \((e, e, x) - 2(e, e, ex) = 0\) to get \(-(e, e, ex) = 0\) and therefore \((e, e, x) = 0\). From \( B(e, e, x) = 0\) we get the rest of (23).

4. Main section. For the following remarks and the lemmas which follow, \( R \) will be assumed to be a simple \((-1, 1)\) ring which is not associative and which is of characteristic not 2 or 3. For Lemmas 4 and 5 \( R \) is also assumed to have an idempotent \( e \neq 0\).

Before proceeding to Lemma 4 we first observe that, when \( R \) has an idempotent \( e \), we have \( R = R_{ij} + R_{io} + R_{oi} + R_{oo} \) where \( R_{ij} = \{x \in R | ex = ix \text{ and } xe = jx\} \). This follows from (23) and the fact that \( x = exe + (ex - exe) + (xe - exe) - (x - ex - xe + exe) \), and the right alternative law.

Lemma 4. The multiplication for the spaces \( R_{ij} \) is as follows:

1. \( R_{ij} \cup R_{ik} \subset R_{ik} \);
2. \( R_{ij} \cup R_{im} = 0 \) for \( j \neq k \) except when \( i = k \) and \( j = m \);
3. \( R_{io} \cup R_{io} \subset R_{ii} \) and \( R_{oi} \cup R_{oo} \subset R_{oo} \). (This differs from the associative situation only in (3)).

Proof. When \( x \in R_{ij}, y \in R_{ji} \), we have directly that \((x, e, y) = (y, e, x) = 0\). From \( B(x, e, y) = 0 \) and \( B(y, e, x) = 0 \) we have \((x, y, e) = (y, x, e) = 0\). Thus from \( A(x, y, e) = 0 \) and \( A(y, x, e) = 0 \) we get \((e, x, y) = 0\). Thus we can easily see that \( R_{ij} \cup R_{ji} \subset R_{ii} \).

Next we take \( x \in R_{ij}, y \in R_{ii} \) where \( i \neq j \). We can compute directly that \((y, e, x) = 0\). From \( K(x, e, y) = -(1)^i(x, e, y) = 0 \) we get \((x, e, y) = 0\). It follows from \( B(x, e, y) = 0, B(y, e, x) = 0, A(x, y, e) = 0, \) and \( A(y, x, e) = 0 \) that \((x, e, y) = (y, x, e) = (e, x, y) = 0\). Thus \( R_{ij} \cup R_{ji} \subset R_{ii} \).

When \( x \in R_{oo} \) and \( y \in R_{oi} \) we have \( K(x, e, y) = -(x, e, y) = 0 \) and thus \( xy = 0 \). We also have \( K(y, e, x) = (y, e, x) = 0 \) and thus \( yx = 0 \). This gives \( R_{oo} \cup R_{oi} \subset R_{ii} \).

When \( x \in R_{oo} \) and \( y \in R_{io} \) we have \((y, e, x) = 0\) by direct computation. Thus \( A(x, e, y) = B(x, e, y) = 0 \) gives \((e, x, y) = 0\). Hence \( e(xy) = xy \). From \( B(y, e, x) = 0 \) and \((y, e, x) = 0 \) we get \((y, x, e) = 0 \) and so \( A(e, x, y) = 0 \) gives \((e, y, x) = 0\). This can be expanded to give \( yx - e(yx) = xy = 0 \). Multiplying this last equation on the left by \( e \) gives \( -e(xy) = 0 \) in view of (23). Hence \( xy = 0 \). Thus \( e(yx) = yx \).
From \((y, x, e) = 0\) we have \((yx)e = 0\). Therefore \(R_{00}R_{10} = 0\) and \(R_{10}R_{00} \subset R_{10}\).

When \(x \in R_{01}\) and \(y \in R_{11}\) we have \((x, e, y) = 0\) by direct calculation. Thus \(B(x, y, e) = 0\) gives \((x, y, e) = 0\) and \((xy)e = xy\). Now \(0 = H(y, e, x)\) gives \((y, e, x)e = 0\). Therefore \((e, x, y) = 0\) along with \((x, y, e) = 0\) gives \((e, x, y)e = 0\). By the remark just before (17) we have \((e, (e, z, w)) = 0\) for all \(z, w \in R\). Hence \(e(e, y, x) = 0\) and \(e(e(xy)) = 0\) and \(e(xy) = 0\) by (23). Therefore \(xy \in R_{01}\). Since \(e(xy) = 0\), \((e, x, y) = 0\). Combining this with \((e, x, y) = 0\) gives \((y, e, x) = 0\) and \(yx = 0\). Thus \(R_{01}R_{11} \subset R_{01}\) and \(R_{11}R_{01} = 0\).

For the final case \(x, y \in R_i\) where \(i \neq j\) we have \(H(x, e, y) = 0\) combined with \((e, (x, e, y)) = 0\) to give \((x, e, y) \in R_{ij}\). But \((x, e, y) = \pm xy\). Hence \(xy \in R_{ii}\).

**Lemma 5.** For all \(x, y \in R\) and the idempotent \(e\) the following identity holds:

\[
(e, x, y) = 0.
\]

**Proof.** For \(x, y \in R\) we have that \(x = x_{11} + x_{10} + x_{01} + x_{00}\) and \(y = y_{11} + y_{10} + y_{01} + y_{00}\) where \(x_{ij}, y_{ij} \in R_{ij}\). Direct calculation and Lemma 4 yield (24).

**Lemma 6.** Every element of \(R\) is the sum of a finite number of associators.

**Proof.** Let \(S\) be the class of elements in \(R\) which are sums of a finite number of associators. Since \(R\) is not associative \(S \neq 0\). We will show that \(S\) is an ideal of \(R\). From \(F(w, x, y, z) + F(w, y, x, z) + B(w, x, y)z = 0\) we have \(w((x, y, z) + (y, x, z)) \in S\) for all \(w, x, y, z \in R\). Therefore \(wA(x, y, z) + wB(x, y, z) + w((x, y, z) + (y, x, z)) - w((x, z, y) + (z, x, y)) - wB(y, x, z) = 3w(x, y, z)\) is an element of \(S\). However \(3w\) is an arbitrary element of \(R\), and so \(RS \subset S\). Next from \(F(w, x, y, z) = 0\) we see that \(SR \subset RS + S \subset S\) and \(S\) is an ideal not zero. Hence \(S = R\).

The center of \(R\) is defined to be the elements \(n \in U\) with the property that \((n, x, y) = (x, n, y) = (x, y, n) = 0\) for all \(x, y \in R\).

**Lemma 7.** Let \(n \in R\) satisfy \((n, x, y) = 0\) for all \(x, y \in R\). Then \(n\) is in the center of \(R\).

**Proof.** We must first show that \(n \in U\). To do this we need only show that \(n\) commutes with any associator because of Lemma 6. \(M(x, n, y) = 0\) gives \((x, (y, n, x)) = 0\) since \((n, y, x) = 0\). Thus from \((x, 0, A(n, x, y)) = 0\) we get \((x, (x, y, n)) = 0\). Combining these results with \(G(x, y, n, x) = 0\) gives \((n, (x, x, y)) = 0\). Replacing \(x\) with \(x + z\)
in this last identity gives \((n, (x, z, y)) + (n, (z, x, y)) = 0\) for all \(x, y, z\)
in \(R\). Thus \(0 = (n, A(x, y, z)) + ((n, (x, y, z)) + (n, (y, x, z))) - ((n, (x, z, y)) + (n, (z, x, y))) + (n, B(x, y, z)) - (n, B(y, z, x)) = 3(n, (x, y, z))\) and \((n, (x, y, z)) = 0\) for any associator \((x, y, z)\) in \(R\).

We now have \(n \in U\) because every element of \(R\) is the sum of associators. Because of (13) we also have \(A(n, x, y) + B(x, y, n) = 2(x, y, n) = 0\). Therefore \((x, y, n) = 0\) and from \(A(n, x, y) = 0\) we have \((y, n, x) = 0\). This shows that \(n\) is in the center of \(R\).

**Theorem.** Let \(R\) be a simple ring of type \((-1, 1)\). Let \(R\) have characteristic not 2 or 3. Let \(e\) be an idempotent of \(R\) such that \(e \neq 0, 1\). Then \(R\) is associative.

**Proof.** Assume \(R\) is not associative. Then Lemma 5 gives \((e, x, y) = 0\) for all \(x, y\) in \(R\). Lemma 7 then gives us that \(e\) is in the center of \(R\). Therefore \(R_{01} = R_{10} = 0\) and \(R = R_{11} + R_{00}\). But then \(R_{00}\) is an ideal of \(R\) by Lemma 4 and \(e \in R_{00}\). Thus \(R_{00} = 0\) and \(R = R_{11}\), a contradiction since \(e\) is the unity element of \(R_{11}\).

5. Example. The following example is given by Albert in [2] as a right alternative algebra. It can be shown to satisfy (1) and (2) hence it is a \((-1, 1)\) algebra. It is finite dimensional and Lemma 3 fails for it.

Let \(F\) be a field and \(A\) an algebra with basis elements \(e, u, v, w, z\). We define multiplication of basis elements as follows: \(e^2 = e, eu = v, ev = 0, ue = u, ew = w - z, ez = ze = z\) and all other products are zero.

**Bibliography**


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