A NOTE ON FINITE GROUPS WITH AN ABELIAN SYLOW GROUP

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It has been conjectured that if the order g of a finite noncyclic simple group G is divisible by a prime power p^n, then $g > p^{2n}$. We shall show this in the case that the p-Sylow groups of G are abelian. In fact, we shall prove the theorem:

THEOREM. *Let G be a finite group. Let p be a prime and assume that the p-Sylow subgroups P of G are abelian. Then the intersection of all p-Sylow subgroups of G appears as intersection of P with one of its conjugates.*

COROLLARY. *If the finite group G of order g has an abelian p-Sylow group P of order p^n, $g \neq p^n$, and if the maximal normal p-subgroup D has order p^d, the number of conjugates of P is at least $p^{n-d}+1$ and $g \geq p^n(p^{n-d}+1)$. In particular, if $D=\{1\}$, $g \geq p^n(p^n+1)$.*

PROOF OF THE THEOREM. If the p-Sylow group P has order 1, the theorem is trivial. We use induction. If $D \neq \{1\}$, we can deduce the statement for G from that for G/D. Hence we may assume $D=\{1\}$. Suppose that

$$P \cap P_1 \cap \cdots \cap P_r = \{1\} \tag{1}$$

where $P_1, P_2, \cdots, P_r$ are conjugates of the p-Sylow subgroup $P=P_0$ of G and where the representation of $\{1\}$ as such an intersection with a minimal r is chosen. If $r=1$, we are finished. Assume then $r \geq 2$ and set

$$M = P_1 \cap P_2 \cap \cdots \cap P_{r-1}, \tag{2}$$

$$T = P \cap M = P \cap P_1 \cap \cdots \cap P_{r-1} \neq \{1\} \tag{3}$$

while by (1)

$$T \cap P_r = \{1\}.$$

Let H be the subgroup generated by $P, P_1, \cdots, P_{r-1}$,

$$H = \{P, P_1, P_2, \cdots, P_{r-1}\}.$$

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Since the P_j are abelian and $T \subseteq P_j$ for $j=0, 1, \cdots, r-1$, T is included in the center $Z(H)$ of H. Hence $H \neq G$, since otherwise T would be normal in G and then T would belong to all p-Sylow subgroups of G, contrary to (1). In particular, the theorem is true for H in the place of G.

The group $T(H \cap P_r)$ is an abelian p-subgroup of H. Let R be a p-Sylow subgroup of H with $R \supseteq T(H \cap P_r)$. By the theorem, applied to H, there exists a p-Sylow subgroup R^* of H such that the intersection X of all the p-Sylow subgroups of H is equal to $R \cap R^*$. The groups $P, P_1, \cdots, P_{r-1}$ are p-Sylow subgroups of H and hence

$$X \subseteq P \cap P_1 \cap \cdots \cap P_{r-1} = T$$

while the p-group $T \subseteq Z(H)$ must belong to all p-Sylow groups of H and hence to X. Thus, $T=X$ and

$$T = R \cap R^*.$$

If $\sigma \in R^* \cap P_r$, then $\sigma \in H \cap P_r \subseteq R$. Hence $\sigma \in R \cap R^* = T$. Consequently, $\sigma \in T \cap P_r = \{1\}$; cf. (3) and (1). This shows that $R^* \cap P_r = \{1\}$. Since $P \subseteq H$, the p-Sylow-groups of H are Sylow groups of G. In particular, R^* is a conjugate of P. Replacing P_r by a suitable conjugate, we obtain a p-Sylow subgroup $P^\sharp$ of P with $P \cap P^\sharp = \{1\}$ and this concludes the proof.

If in the notation of the theorem, we have $D = P \cap P^*$ where P^* is a conjugate of P, then as is well known, we have p^{n-d} distinct p-Sylow groups of the form $\sigma^{-1}P\sigma$ with $\sigma \in P^*$. Since they are all different from P^*, the number of conjugates of P is at least $p^{n-d}+1$. On the other hand, the number of conjugates is g/m where m is the order of the normalizer of P in G. Since $m \geqq p^n$, $g \geqq p^n(p^{n-d}+1)$. This establishes Corollary 1.

Corollary 2. *If the finite group G of order g has an abelian p-Sylow group P of order p^n, if G possesses neither a normal p-subgroup different from $\{1\}$ nor a normal subgroup of index p, then $g \geqq 2p^n(p^n+1)$.*

Indeed, it follows from Burnside's theorem that the normalizer of P has at least order $2p^n$ in this case while the number of p-Sylow groups is at least p^n+1.

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