A GENERALISATION OF BELLMAN'S LEMMA

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1. Certain integral inequalities occur very frequently in the theory of ordinary differential equations in various contexts. The aim of this note is to prove a general theorem which is a generalisation of Bellman's Lemma [1], and which also includes other generalisations of the same. We shall also indicate its applications to some problems in ordinary differential equations. We shall first prove the following

THEOREM 1. *If $\phi(x) \leqq \eta + \int_{x_0}^{x} f(s, \phi(s))ds$ where $f(x, y)$ is continuous and monotonic increasing in y in the region R defined by $|x - x_0| \leqq a$; $|y - \eta| \leqq b$, where a and b are positive real numbers; and $\phi(x)$ is continuous in the interval $|x - x_0| \leqq a$, then $\phi(x) \leqq \chi(x)$ where $\chi(x)$ is the maximal solution of the differential equation $z' = f(x, z)$ through (x_0, η) for $x \geqq x_0$. (We shall call this differential equation the associated differential equation of the above integral inequality.)*

PROOF. Take $\phi(x)$ as the zero approximation of the solution of the differential equation $z' = f(x, z)$ through (x_0, η) and set up the successive approximations recursively by

$$\phi_{k+1}^{(x)} = \eta + \int_{x_0}^{x} f(s, \phi_k(s))ds.$$

These successive approximations exist at least on the interval $|x - x_0| \leqq \alpha$ where $\alpha = \min (a, b/M)$ where M is a positive number such that $|f(x, y)| \leqq M$. Further, this sequence of successive approximations is equicontinuous in this interval, for,

$$\begin{aligned} |\phi_n(x_1) - \phi_n(x_2)| &= \left| \int_{x_1}^{x_2} f(s, \phi_{n-1}(s))ds \right| \leqq \int_{x_1}^{x_2} |f(s, \phi_{n-1}(s))| ds \\ &\leqq |x_1 - x_2| M \leqq \epsilon \qquad \text{if } |x_1 - x_2| \leqq \epsilon/M = \delta. \end{aligned}$$

It is further uniformly bounded because

$$|\phi_n(x)| \leqq |\eta| + M|x_2 - x_1| \leqq \eta + M\alpha.$$

We can show by induction that these successive approximations form a monotonic increasing sequence, for, suppose that $\phi_k^{(x)} \geqq \phi_{k-1}^{(x)}$. Then

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$$\phi_{k+1}^{(x)} - \phi_k^{(x)} = \int_{x_0}^{x} \{f(s, \phi_k^{(s)}) - f(s, \phi_{k-1}^{(s)})\} ds \geqq 0$$

since $f(x, z)$ is monotonic increasing in z. Therefore $\phi_{k+1}^{(s)} \geqq \phi_k^{(x)}$. But the basic hypothesis of our theorem is that the zero approximation $\leqq$ first approximation. So the successive approximations form a monotonic increasing, equi continuous, and uniformly bounded function sequence in the interval $|x-x| \leqq \alpha$, and therefore must converge uniformly to a function $\psi(x)$. Further, it is clear that $\psi(x)$ is a solution of the associated differential equation through (x_0, η) and

$$\phi(x) \leqq \psi(x) \qquad \text{for } x_0 \leqq x \leqq x + \alpha.$$

Therefore

$$\phi(x) \leqq \chi(x) \qquad \text{for } x_0 \leqq x \leqq x + \alpha$$

when $\chi(x)$ is the maximal solution through (x_0, η).

As a counterpart to this theorem we can similarly prove the following

THEOREM 2. *Under the same conditions as in Theorem* 1 *if*

$$\phi(x) \geqq \eta + \int_{x_0}^{x} f(s, \phi(s)) ds$$

then $\phi(x) \geqq$ *minimal solution of the associated differential equation through* (x_0, η) *for* $x_0 \leqq x \leqq x+\alpha$.

The proof is the same except that in this case the successive approximations form a monotonic decreasing sequence converging to a solution of the associated equation.

The following may be obtained as corollaries to the above theorems.

COROLLARY 1. *Under the condition of Theorem* 1 *if*

$$\phi(x) \leqq \psi(x) + \int_{x_0}^{x} f(s, \phi(s)) ds$$

then $\phi(x) \leqq \psi(x) + \chi(x)$ *for* $x \geqq x_0$ *where* $\chi(x)$ *is the maximal solution of* $z' = f(x, z + \psi(x))$ *through* $(x_0, 0)$ *as far as this maximal solution exists.*

PROOF. Put $r(x) = \phi(x) - \psi(x)$ and the inequality becomes

$$r(x) \leqq \int_{x_0}^{x} f(s, r^{(s)} + \psi(s)) ds.$$

Apply Theorem 1, and we obtain $r(x) \leqq \chi(x)$. Therefore $\phi(x) \leqq \psi(x) + \chi(x)$. The counterpart to this may be stated as

COROLLARY 2. *Under the conditions of the above theorem if*

$$\phi(x) \geqq \psi(x) + \int_{x_0}^{x} f(s, \phi(s))ds$$

then $\phi(x) \geqq \psi(x) + \chi(x)$ *for* $x \geqq x_0$ *when* $\chi(x)$ *is the minimal solution of the associated equation in Corollary* 1.

Similar theorems may also be proved for intervals with x_0 as the right end point.

2. A very special case of Theorem 1 is what is known as Bellman's Lemma [1] which is as follows:

$$|y(x)| \leqq M + \int_0^x |f(s)| \cdot |y(s)| ds$$

then

$$|y(x)| \leqq M \exp \int_0^x |f(t)| dt.$$

This is obtained by putting $f(x, y) = |f(x)| y$, $x_0 = 0$ and $\eta = M$ in Theorem 1.

Another special case of the same theorem is obtained by putting $f(x, y) = v(x) \cdot g(y)$ where $v(x)$ is non-negative and $g(y)$ is monotonic increasing in y. This case is considered in [2] and also in [3]. It is not necessary to work through the details to prove results of [2] and [3] from Theorem 1.

3. It is clear that in many situations in ordinary differential equations where we use a Lipschitz condition or Lipschitz-like condition we may obtain more general results by applying the above theorems. For example [4] contains the following proposition on approximate solutions.

Suppose f satisfies a Lipschitz condition with Lipschitz constant k; ϕ_1 and ϕ_2 are ϵ_1 and ϵ_2 approximate solutions of the differential equation $x' = f(t, x)$ and for some τ we have $|\phi_1(\tau) - \phi_2(\tau)| \leqq \delta$. Then for $t \geqq \tau$ we have

$$|\phi_1(t) - \phi_2(t)| \leqq \delta e^{k(t-\tau)} + \frac{\epsilon}{k}(e^{k(t-\tau)} - 1) \qquad \text{where } \epsilon = \epsilon_1 + \epsilon_2.$$

If f satisfies the more general condition

$$|f(x, y_1) - f(x, y_2)| \leqq \omega(x, |y_1 - y_2|)$$

where $\omega(x, z)$ satisfies conditions of Theorem 1 we may show easily, applying Theorem 1, that $|\phi_1(t)-\phi_2(t)| \leqq \chi(t)$ for $t \geqq \tau$ where $\chi(t)$ is the maximal solution of $z' = \omega(t, z) + \epsilon$ through (τ, δ). Further the first few examples of page 37 of [4] can all be solved by the application of Corollary 1. Similarly Theorems 1 and 2 can be used in a natural way to extend the results of [3] concerning bounds on the norm of a solution of a differential equation.

References

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THE G-FUNCTIONS AS UNSYMMETRICAL FOURIER KERNELS. II

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1. A function $K(x)$ by means of which an arbitrary function $f(x)$ subject to appropriate conditions, is capable of being represented as a repeated integral of the form

$$f(x) = \int_0^\infty K(xu) \int_0^\infty K(uy) f(y) dy du \tag{1.1}$$

has been called a *Fourier kernel* by Hardy and Titchmarsh [1, p. 116]. This is a symmetrical formula. There are also unsymmetrical formulae of the type

$$f(x) = \int_0^\infty K(xu) \int_0^\infty H(uy) f(y) dy du \tag{1.2}$$

in which the kernels in the two integrals are different functions. If $f(t)$ is not continuous at $t = x$, $f(x)$ on the left-hand side of (1.1) or (1.2) is replaced by

$$\frac{1}{2}\{f(x+0) + f(x-0)\}.$$

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