ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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In this paper we study the behavior of solutions of the differential equation

$$u'' = (f(t) + g(t))u \tag{1}$$

as $t \to \infty$. We assume that the general solution of

$$z'' = f(t)z \tag{2}$$

is known, and prove the following theorem.

THEOREM. *Let z_1 and z_2 be independent solutions of* (2), *let g be continuous in $0 \leqq t < \infty$, and assume that*

$$\int_0^\infty |g| y dt < \infty, \tag{3}$$

where

$$y(t) = \max[|z_1(t)|^2, |z_2(t)|^2].$$

Then, if a and b are arbitrary constants, there is a solution of (1) *which can be written in the form*

$$u = \alpha(t)z_1 + \beta(t)z_2,$$

with $\lim_{t\to\infty} \alpha(t) = a$ *and* $\lim_{t\to\infty} \beta(t) = b$.

In the proof of this theorem, we use a lemma due to Bellman [1, p. 35].

LEMMA. *If $u, v \geqq 0$, if c_1 is a positive constant, and if*

$$u \leqq c_1 + \int_0^t uv dt_1$$

then

$$u \leqq c_1 \exp\left(\int_0^t v dt_1\right).$$

PROOF OF THE THEOREM. We write

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$$u = Az_1 + Bz_2 \tag{4}$$

and require that

$$A'z_1 + B'z_2 = 0. \tag{5}$$

Differentiating (4) twice yields

$$u'' = Az_1'' + Bz_2'' + A'z_1' + B'z_2'.$$

From (2) and (4), the sum of the first two terms is $f(t)u$. Hence, from (1) and (4),

$$A'z_1' + B'z_2' = g(Az_1 + Bz_2). \tag{6}$$

Equations (5) and (6) can be solved to yield

$$\begin{aligned} A' &= \frac{gz_2}{W}(Az_1 + Bz_2), \\ B' &= \frac{-gz_1}{W}(Az_1 + Bz_2), \end{aligned} \tag{7}$$

where $W = z_1'z_2 - z_1z_2' =$constant. Integrate (7) to obtain

$$A(t) = A(0) + \frac{1}{W}\int_0^t gz_2(Az_1 + Bz_2)dx, \tag{8}$$

$$B(t) = B(0) - \frac{1}{W}\int_0^t gz_1(Az_1 + Bz_2)dx \tag{9}$$

for $t \geqq 0$. From this it follows that

$$\begin{aligned} |A(t)| + |B(t)| \leqq\ & |A(0)| + |B(0)| \\ & + \frac{2}{|W|}\int_0^t |g|\, y(|A| + |B|)dx. \end{aligned}$$

From Bellman's lemma

$$|A(t)| + |B(t)| \leqq (|A(0)| + |B(0)|)\exp\frac{2}{|W|}\int_0^t |g|\, ydx$$

for positive t, and therefore, from (3), A and B are bounded. Now it follows that the integrals on the right sides of (8) and (9) are convergent at infinity, so that $\lim_{t\to\infty} A(t)$ and $\lim_{t\to\infty} B(t)$ exist and are finite.

Let $A_1(t)$ and $B_1(t)$, and $A_2(t)$ and $B_2(t)$ be the solutions of (7) such that $A_1(0) = 1$, $B_1(0) = 0$, $A_2(0) = 0$, $B_2(0) = 1$. Denote $\lim_{t\to\infty} A_i(t) = a_i$, and $\lim_{t\to\infty} B_i(t) = b_i$ $(i = 1, 2)$. We have shown above that these limits exist. The function $A_1B_2-A_2B_1$ is a constant (unity), since its derivative vanishes identically, as may be seen from (7). Hence $a_1b_2-a_2b_1=1$, and it is easy to verify that $\alpha(t)$ and $\beta(t)$, defined by

$$\alpha(t) = (b_2a - a_2b)A_1(t) + (ba_1 - ab_1)A_2(t),$$
$$\beta(t) = (b_2a - a_2b)B_1(t) + (ba_1 - ab_1)B_2(t),$$

satisfy the requirements of the theorem.

This theorem contains a previously known result [1, p. 112] to the effect that, if every solution of (2) is bounded, and if $\int_0^\infty |g| dt<\infty$, then every solution of (1) is bounded. It also contains results of Fubini [2], who studied the special cases $f(t)=1$ and $f(t)=-1$.

For the special case $f(t)\equiv 0$, one can conclude from the present theorem that, if $\int_0^\infty |g| t^2dt<\infty$, then every solution of $u''=g(t)u$ can be written in the form $u=A(t)+B(t)t$, where A and B approach prescribed limits. This in turn contains a result of Sansone [3], demonstrated under the assumption that $|g(t)|<kt^{-c}$, $k>0$, $c>3$.

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REFERENCES

1. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York, 1953.

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3. G. Sansone, *Equazioni differenziali nel campo reala*, 2nd ed., Nicola Zanichelli, Bologna, 1948.

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