ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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In this paper we study the behavior of solutions of the differential equation

\( u'' = (f(t) + g(t))u \)

as \( t \to \infty \). We assume that the general solution of

\( z'' = f(t)z \)

is known, and prove the following theorem.

**Theorem.** Let \( z_1 \) and \( z_2 \) be independent solutions of (2), let \( g \) be continuous in \( 0 \leq t < \infty \), and assume that

\[
\int_0^\infty |g| y(t) dt < \infty,
\]

where

\[
y(t) = \max [ |z_1(t)|^2, |z_2(t)|^2 ].
\]

Then, if \( a \) and \( b \) are arbitrary constants, there is a solution of (1) which can be written in the form

\[
u = a(t)z_1 + b(t)z_2,
\]

with \( \lim_{t \to \infty} \alpha(t) = a \) and \( \lim_{t \to \infty} \beta(t) = b \).

In the proof of this theorem, we use a lemma due to Bellman [1, p. 35].

**Lemma.** If \( u, v \geq 0 \), if \( c_1 \) is a positive constant, and if

\[
u \leq c_1 + \int_0^t uvd\tau,
\]

then

\[
u \leq c_1 \exp \left( \int_0^t \nu d\tau \right).
\]

**Proof of the Theorem.** We write
(4) \[ u = A z_1 + B z_2 \]
and require that
(5) \[ A' z_1 + B' z_2 = 0. \]
Differentiating (4) twice yields
\[ u'' = A z_1'' + B z_2'' + A' z_1' + B' z_2'. \]
From (2) and (4), the sum of the first two terms is \( f(t) u \). Hence, from (1) and (4),
(6) \[ A' z_1' + B' z_2' = g(A z_1 + B z_2). \]
Equations (5) and (6) can be solved to yield
\[ A' = \frac{g z_2}{W} (A z_1 + B z_2), \]
(7) \[ B' = \frac{-g z_1}{W} (A z_1 + B z_2), \]
where \( W = z_1' z_2 - z_2' z_1 = \text{constant} \). Integrate (7) to obtain
(8) \[ A(t) = A(0) + \frac{1}{W} \int_0^t g z_2 (A z_1 + B z_2) \, dx, \]
(9) \[ B(t) = B(0) - \frac{1}{W} \int_0^t g z_1 (A z_1 + B z_2) \, dx \]
for \( t \geq 0 \). From this it follows that
\[ | A(t) | + | B(t) | \leq | A(0) | + | B(0) | + \frac{2}{| W |} \int_0^t | g | | y A | + | B | \, dx. \]
From Bellman's lemma
\[ | A(t) | + | B(t) | \leq (| A(0) | + | B(0) |) \exp \frac{2}{| W |} \int_0^t | g | \, y dx \]
for positive \( t \), and therefore, from (3), \( A \) and \( B \) are bounded. Now it follows that the integrals on the right sides of (8) and (9) are convergent at infinity, so that \( \lim_{t \to \infty} A(t) \) and \( \lim_{t \to \infty} B(t) \) exist and are finite.

Let \( A_1(t) \) and \( B_1(t) \), and \( A_2(t) \) and \( B_2(t) \) be the solutions of (7) such that \( A_1(0) = 1, B_1(0) = 0, A_2(0) = 0, B_2(0) = 1 \). Denote \( \lim_{t \to \infty} A_i(t) = a_i, \) and \( \lim_{t \to \infty} B_i(t) = b_i \) \( (i = 1, 2) \). We have shown above that these
limits exist. The function \( A_1B_2 - A_2B_1 \) is a constant (unity), since its derivative vanishes identically, as may be seen from (7). Hence \( a_1b_2 - a_2b_1 = 1 \), and it is easy to verify that \( \alpha(t) \) and \( \beta(t) \), defined by

\[
\alpha(t) = (b_2a - a_2b)A_1(t) + (ba_1 - ab_1)A_2(t),
\]

\[
\beta(t) = (b_2a - a_2b)B_1(t) + (ba_1 - ab_1)B_2(t),
\]

satisfy the requirements of the theorem.

This theorem contains a previously known result [1, p. 112] to the effect that, if every solution of (2) is bounded, and if \( \int_{\infty}^{\infty} |g| dt < \infty \), then every solution of (1) is bounded. It also contains results of Fubini [2], who studied the special cases \( f(t) = 1 \) and \( f(t) = -1 \).

For the special case \( f(t) = 0 \), one can conclude from the present theorem that, if \( \int_{\infty}^{\infty} |g| t^2 dt < \infty \), then every solution of \( u'' = g(t)u \) can be written in the form \( u = A(t) + B(t)t \), where \( A \) and \( B \) approach prescribed limits. This in turn contains a result of Sansone [3], demonstrated under the assumption that \( |g(t)| < kt^{-c}, k > 0, c > 3 \).

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References


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