SOME GENERAL FORMULAS FOR DOUBLE LAPLACE TRANSFORMATIONS

GERALDINE A. COON AND DOROTHY L. BERNSTEIN[1]

In many problems amenable to double Laplace transformations it is necessary to perform certain operations upon the functions involved. This paper studies the effects of such operations upon the transforms. The one- and two-dimensional transforms are defined by $L_s[H]=\int_0^\infty e^{-sx}H(x)dx$ and $L[F]=\int_Q e^{-sx-tv}F(x,y)d(x,y)$ where Q is the open first quadrant and s and t are complex variables: $s=\sigma+i\lambda$, $t=\tau+i\mu$. The transforms are denoted by $h(s)$ and $f(s, t)$. It is assumed that $L[F]$ converges boundedly rather than absolutely; various types of convergence are defined in [1].

1. **Derivatives and integrals.** This section presents theorems for the Laplace transforms of functions $H^{(\alpha)}(x)$ and $F_{\alpha\beta}(x, y)$ defined in [2]; these include pure derivatives and integrals as well as operations such as $(\partial/\partial x)\int_0^y F(x, v)dv$. The hypotheses are more general than those customarily given. H and F are taken to be functions of class A_m. The definitions and basic properties of this class are given in [2]. In brief, functions of class A_0 are defined on measurable kernels of $I=(0, \infty)$ or of $Q=I\times I$ and are summable in finite intervals or rectangles. For $m\geqq 1$, a function of class A_m is defined on I or Q and its derivatives of order $m-1$ are absolutely continuous in finite intervals or rectangles, with proper attention to boundary values.

THEOREM 1. *Let $H(x)$ be a function of class A_m with $m\geqq 0$ and let one of the following be true for a given $\sigma_0\geqq 0$:*

(i) *$L_s[H^{(m)}]$ converges at $s=s_0$,*

(ii) *$|H^{(m-1)}(x)|\leqq Me^{\sigma_0 x}$ for all $x>0$,*

(iii) *$\left|\int_0^X e^{-s_1 x}H^{(m)}(x)dx\right|\leqq Me^{aX}$ for all $X>0$ and $\sigma_0=\sigma_1+a$. Then for $\sigma>\sigma_0$, $L_s[H^{(m)}]$ converges, $L_s[H^{(\alpha)}]$ converges absolutely for all $-\infty<\alpha<m$, and*

$$L_s[H^{(\alpha)}] = s^\alpha h(s) - \sum_{r=0}^{\alpha-1} c_r s^{\alpha-1-r} \qquad (\alpha \leqq m), \tag{1.1}$$

where $c_r=H^{(r)}(+0)$. If b is any constant greater than σ_0,

$$|H^{(\alpha)}(x)| \leqq Me^{bx} \text{ for all } x > 0 \qquad (\alpha < m). \tag{1.2}$$

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(*When* $\alpha \leqq 0$, *the sum in* (1.1) *is replaced by zero.*)

PROOF. From Theorem 5 of [2] $H^{(\alpha)}$ exists on I for $\alpha < m$ and almost everywhere on I for $\alpha = m$; also, $H \in A_k$ for all $0 \leqq k \leqq m$. Let $m \geqq 1$. Since $H^{(m-1)} \in A_1$, apply Theorem 3.1 of [1]: for $\sigma > \sigma_0$, $L_s[H^{(m)}]$ converges, $L_s[H^{(m-1)}]$ converges absolutely, $L_s[H^{(m)}] = sL_s[H^{(m-1)}] - c_{m-1}$, and $|H^{(m-1)}(x)| \leqq Me^{bx}$ where $b > \sigma_0$. Since $H \in A_{m-1}$, $H^{(m-2)} \in A_1$. The absolute convergence of $L_s[H^{(m-1)}]$ allows us to apply Theorem 3.1 of [1] to $H^{(m-2)}$ under hypothesis (ii). This yields the absolute convergence of $L_s[H^{(m-2)}]$ for $\sigma > \sigma_0$, the formula $L_s[H^{(m-1)}] = sL_s[H^{(m-2)}] - c_{m-2}$, and the fact that $H^{(m-2)}(x)$ is of exponential type. Continuing in this manner gives the theorem for $0 \leqq \alpha \leqq m$.

By (1.2), since $H \in A_0$, $H(x)$ and hence its integral $H^{(-1)}(x)$ is of exponential type. The theorem for the case $\alpha = -1$ follows from the remark after Corollary 2 of Theorem 2.8 of [1]. By induction the theorem holds for any negative integer α.

The case $m = 0$ is easily proved.

Turning now to the two-dimensional case, we use the following notation for the boundary values:

$$(1.3) \quad H_{\alpha\beta}(x) = F_{\alpha\beta}(x, +0), \quad G_{\alpha\beta}(y) = F_{\alpha\beta}(+0, y), \quad c_{\alpha\beta} = F_{\alpha\beta}(+0, +0).$$

It was shown in Theorems 1 and 2 of [2] that if $F \in A_m$, then $F \in A_k$ for $0 \leqq k \leqq m$ and the following quantities exist at least almost everywhere: $F_{\alpha\beta}(x, y)$ for $\alpha, \beta, \alpha+\beta \leqq m$, $G_{\alpha\beta}(y)$ for $\alpha < m$, $\beta \leqq m$, $\alpha+\beta < m$, $H_{\alpha\beta}(x)$ for $\alpha \leqq m$, $\beta < m$, $\alpha+\beta < m$, $c_{\alpha\beta}$ for $\alpha, \beta, \alpha+\beta < m$.

LEMMA. *Let* $R \in A_0$ *and let* $0 \leqq \sigma_0 = \sigma_1 + a$, $0 \leqq \tau_0 = \tau_1 + b$. *If*

$$(1.4) \quad \left| \int_{(0,0)}^{(X,Y)} e^{-s_1 x - t_1 y} R(x, y) d(x, y) \right| \leqq Me^{aX+bY} \quad \textit{for } (X, Y) \in Q,$$

then for $\sigma > \sigma_0$, $\tau > \tau_0$ *and for negative* i *and* j, $L[R]$ *converges boundedly,* $L[R_{0j}]$ *converges s-boundedly,* $L[R_{i0}]$ *converges t-boundedly, and* $L[R_{ij}]$ *converges regularly. Also,*

$$(1.5) \quad L[R_{ij}] = s^i t^j r(s, t) \qquad (i, j \leqq 0).$$

PROOF. By Theorem 2.9 of [1] the lemma is true for $i = j = -1$ and (1.5) holds for $-1 \leqq i$, $j \leqq 0$. Induction gives the general result.

THEOREM 2. *Let* $F \in A_m$ *and* $\sigma_0 \geqq 0$, $\tau_0 \geqq 0$. *Suppose that one of the following holds for all* $0 \leqq p$, $0 \leqq q$, $p+q = m$:

(i) $L[F_{pq}]$ *converges boundedly at* (s_0, t_0).

(ii) $\left| \int_{(0,0)}^{(X,Y)} e^{-s_1 x - t_1 y} F_{pq}(x, y) d(x, y) \right| \leqq Me^{aX+bY}$ *for* $(X, Y) \in Q$ *and* $\sigma_0 = \sigma_1 + a$, $\tau_0 = \tau_1 + b$.

Then the following statements are valid for $\sigma > \sigma_0$, $\tau > \tau_0$:

(1.6) *For* $\alpha+\beta<m$, $-\infty<\alpha<m$, $-\infty<\beta\leqq m$, $L_t[G_{\alpha\beta}]$ *converges and*

$$L_t[G_{\alpha\beta}] = t^{\beta}g_{\alpha 0}(t) - \sum_{k=0}^{\beta-1} c_{\alpha k}t^{\beta-k-1}.$$

(1.7) *For* $\alpha+\beta<m$, $-\infty<\alpha\leqq m$, $-\infty<\beta<m$, $L_s[H_{\alpha\beta}]$ *converges and*

$$L_s[H_{\alpha\beta}] = s^{\alpha}h_{0\beta}(s) - \sum_{k=0}^{\alpha-1} c_{k\beta}s^{\alpha-k-1}.$$

(1.8) *For* $\alpha+\beta\leqq m$, $\alpha\leqq m$, $\beta\leqq m$, $L[F_{\alpha\beta}]$ *converges boundedly and*

$$L[F_{\alpha\beta}] = s^{\alpha}t^{\beta}f(s,t) - s^{\alpha}\sum_{k=0}^{\beta-1} t^{\beta-k-1}h_{0k}(s) - t^{\beta}\sum_{d=0}^{\alpha-1} s^{\alpha-d-1}g_{d0}(t) + \sum_{k=0}^{\beta-1}\sum_{d=0}^{\alpha-1} s^{\alpha-d-1}t^{\beta-k-1}c_{dk}.$$

(1.9) *The convergence of* $L_t[G_{\alpha\beta}]$, $L_s[H_{\alpha\beta}]$, *and* $L[F_{\alpha\beta}]$ *is regular for* $\alpha<m$, $\beta<m$, $\alpha+\beta<m-1$. *The convergence of* $L[F_{\alpha\beta}]$ *is s-bounded for* $\alpha=m$, $\beta<0$ *or for* $0\leqq\alpha$, $0\leqq\beta=m-1-\alpha$. *The convergence is t-bounded for* $\beta=m$, $\alpha<0$ *or for* $0\leqq\alpha$, $0\leqq\beta=m-1-\alpha$. (*The sums vanish when the upper limits are negative.*)

PROOF. It is sufficient to prove the theorem under (ii). For $m=0$ the theorem follows from the lemma by setting $R=F$, $i=\alpha$, $j=\beta$ and remarking that the relevant $G_{\alpha\beta}$, $H_{\alpha\beta}$, and $c_{\alpha\beta}$ vanish.

Let $m\geqq 1$. Theorem 3.3 of [1] can be applied to $\overline{F}=F_{\alpha\beta}$ when $0\leqq\alpha,\beta$; $\alpha+\beta=m-1$, since $\overline{F}_x$ and $\overline{F}_y$ are mth order derivatives of F. Thus $L_s[H_{\alpha\beta}]$ and $L_t[G_{\alpha\beta}]$ converge for $\sigma>\sigma_0$, $\tau>\tau_0$. When $0\leqq\beta<m$ and $\alpha\leqq m-1-\beta$, $H_{0\beta}\in A_{m-1-\beta}$ and $H_{\alpha\beta}=H_{0\beta}^{(\alpha)}$ a.e. by Theorem 2 of [2]. But $L_s[H_{0\beta}^{(m-1-\beta)}]=L_s[H_{m-1-\beta,\beta}]$ converges for $\sigma>\sigma_0$ by the above, which is hypothesis (i) of Theorem 1 for $H_{0\beta}$. Thus $L_s[H_{\alpha\beta}] = L_s[H_{0\beta}^{(\alpha)}]$ converges regularly for $\alpha<m-1-\beta$. Also, $c_{\alpha\beta} =\lim_{x\to 0+} H_{0\beta}^{(\alpha)}(x)$ by Theorem 2 of [2] so that (1.7) follows from (1.1). When $\beta<0$ and $\alpha\leqq m$, $H_{\alpha\beta}=0$ at least a.e.; thus $L_s[H_{\alpha\beta}]$ converges regularly for $\sigma>\sigma_0$ and (1.7) holds. The statements of the theorem concerning $H_{\alpha\beta}$ have been established. A similar argument may be made for $G_{\alpha\beta}$.

From Theorem 3 of [2]

$$\begin{aligned} F_{\alpha\beta}(x,y) &= [F_{pq}(x,y)]_{ij} + \sum_{k=0}^{-i-1} G_{\alpha+k,\beta}(y)\frac{x^k}{k!} + \sum_{d=0}^{-j-1} H_{\alpha,\beta+d}(x)\frac{y^d}{d!} \\ &\quad - \sum_{k=0}^{-i-1}\sum_{d=0}^{-j-1} c_{\alpha+k,\beta+d}\frac{x^k}{k!}\frac{y^d}{d!} \end{aligned} \tag{1.10}$$

for $\alpha, \beta, \alpha+\beta \leqq m$; $0 \leqq p, q$; $p+q=m$; $i=\alpha-p \leqq 0$; $j=\beta-q \leqq 0$. Apply the lemma to $R=F_{pq}$. Then (1.5) holds for $i, j \leqq 0$.

Formula (1.10) holds in particular for $\alpha=\beta=i+p=j+q=0$. Each term on the right-hand side of the resulting formula has a boundedly convergent transform for $\sigma>\sigma_0$, $\tau>\tau_0$. On substituting this expression for $r(s, t)$ into (1.5), we have

$$
\begin{aligned}
L[R_{ij}] &= s^{p+i}t^{q+j}f(s,t) - s^{p+i}\sum_{d=0}^{q-1} h_{0d}(s)t^{q+j-d-1} - t^{q+j}\sum_{k=0}^{p-1} g_{k0}(t)s^{p+i-k-1} \\
&\quad + \sum_{k=0}^{p-1}\sum_{d=0}^{q-1} c_{kd}s^{p+i-k-1}t^{q+j-d-1}
\end{aligned}
\tag{1.11}
$$

for $\sigma>\sigma_0$, $\tau>\tau_0$, when $i, j \leqq 0$; $0 \leqq p, q$; $p+q=m$.

To establish the results relating to $L[F_{\alpha\beta}]$, we will divide the admissible α and β into six cases. In each case we will find p, q, i, j satisfying the inequalities for which (1.10) holds. Transforming (1.10) gives the desired results.

CASE 1. $\alpha+\beta=m$; $0 \leqq \alpha, \beta$. Take $p=\alpha$, $q=\beta$, $i=j=0$. Since the sums in (1.10) vanish, $L[F_{\alpha\beta}]=L[R]=r(s, t)$. Then (1.11) yields (1.8).

CASE 2. $\alpha+\beta<m$, $\alpha<0$, $\beta=m$. Take $p=j=0$, $q=m$, $i=\alpha$. The last two sums in (1.10) drop out; in the first sum $\alpha+k \leqq \alpha-i-1<0$ so that $G_{\alpha+k,\beta}(y)=0$ a.e. Hence, $L[F_{\alpha\beta}]=L[R_{i0}]$ where the convergence is t-bounded by the lemma. (1.8) then follows from (1.11).

CASE 3. $\alpha+\beta<m$, $\alpha=m$, $\beta<0$. Take $p=m$, $q=i=0$, $j=\beta$. The argument is similar to Case 2, the convergence being s-bounded.

CASE 4. $\alpha+\beta=m-1$; $\alpha, \beta<m$. Either $i=0, j=-1$ or $i=-1, j=0$. In the first instance take $p=\alpha$, $q=\beta+1$ so that (1.10) becomes $F_{\alpha\beta}=[F_{pq}]_{0,-1}+H_{\alpha\beta}$. The convergence of $L[R_{0,-1}]$ is s-bounded. $L_s[H_{\alpha\beta}]$ exists by (1.7), and the formula in (1.8) follows from (1.11) and (1.7). A similar argument holds when $i=-1$, $j=0$, the convergence being t-bounded.

CASE 5. $\alpha+\beta<m-1$, $\alpha<m$, $0 \leqq \beta<m$. Take $p=m-1-\beta$, $q=\beta+1$, $i=\alpha-p$, $j=-1$. $L[F_{\alpha\beta}]$ is obtained from (1.10) by using (1.6)(1.7)(1.11). The convergence of $L[R_{ij}]$ is regular for $i, j<0$; the one-dimensional transforms converge regularly and, hence, so does $L[F_{\alpha\beta}]$.

CASE 6. $\alpha+\beta<m-1$, $\alpha<m$, $\beta<0$. Take $p=m$, $q=0$, $i=\alpha-m$, $j=\beta$. Again $L[R_{ij}]$ converges regularly. Since $\alpha+k \leqq m-1$, $\beta<0$, $\alpha+k+\beta<m-1$, $L_t[G_{\alpha+k,\beta}]$ converges regularly and so does the transform of the first sum of (1.10). In the second sum $\beta+d \leqq \beta-j-1<0$, $\alpha<m$, $\alpha+\beta+d<m-1$ so that $H_{\alpha,\beta+d}$ vanishes a.e. In the third sum $c_{\alpha+k,\beta+d}=0$. Hence, $L[F_{\alpha\beta}]$ converges regularly; formula (1.8) follows from (1.6)(1.11).

2. **Plane waves.** In the solution of hyperbolic differential equations one encounters functions of the form $J(bx+ay)$. We give a comprehensive result for the two-dimensional transform of such a function which holds for all possible signs of a and b. (The function J_b is defined by: $J_b(z)=J(bz)$ for all z.)

THEOREM 3. *Let H and G be functions of class A_0 on I. Set $J(z)=H(z)$ for $z>0$, $J(0)=0$, $J(z)=G(-z)$ for $z<0$ and let $W(x, y)=J(bx+ay)$ for any real constants a and b, not both zero. If $L_s[H]$ and $L_t[G]$ converge for $\sigma>\sigma_0\geqq 0$, $\tau>\tau_0\geqq 0$, then $L[W]$ converges boundedly in R: $\sigma>|b|k_b$, $\tau>|a|k_a$ where $k_a=\sigma_0$ if $a\geqq 0$, $k_a=\tau_0$ if $a<0$. For $ab\leqq 0$*

$$w(s, t) = \frac{aL_t[J_a] - bL_s[J_b]}{as - bt}. \tag{2.1}$$

For $ab>0$ formula (2.1) *holds when $as-bt\neq 0$ and*

$$w\left(\frac{b}{a}t, t\right) = -\frac{a}{b}\frac{d}{dt}L_t[J_a]. \tag{2.2}$$

PROOF. Let $a, b>0$. Set $V(x, y)=J(x+y)$ for $(x, y)\in Q$. By Theorem 6.3 of [1], $L[V]$ converges boundedly for $\sigma>\sigma_0$, $\tau>\sigma_0$ to

$$v(s, t) = [h(t) - h(s)]/(s - t) \text{ for } s - t \neq 0,\ v(t, t) = -h'(t). \tag{2.3}$$

Since $W(x, y)=V(bx, ay)$, a scale change gives $w(s, t)=v(s/b, t/a)/ab$ in the region $\sigma>b\sigma_0$, $\tau>a\sigma_0$. The theorem follows on substituting from (2.3) and noting that $h(t/a)=aL_t[J_a]$.

Let $a>0$, $b<0$. Set $P(x, y)=aP_1(x, y)-bP_2(x, y)$ where

$$P_1(x, y) = (1/a)H(y + bx/a) \text{ and } P_2(x, y) = 0 \qquad \text{for } y + bx/a > 0,$$

$$P_1(x, y) = 0 \text{ and } P_2(x, y) = -(1/b)G(-y - bx/a) \text{ for } y + bx/a < 0.$$

The results on p. 164 of [1] yield $p(s,t)=a[h(t)+g(-as/b)]/(as-bt)$ in the region $\sigma>-b\tau_0/a$, $\tau>\sigma_0$. But $W(x, y)=P(ax, ay)$ so that

$$w(s, t) = p(s/a, t/a)/a^2 \qquad \text{for } \sigma > -b\tau_0,\ \tau > a\sigma_0.$$

Now (2.1) follows easily.

The other cases can be proved in a similar fashion.

THEOREM 4. *Let $V(x, y)=x^m y^n J(bx+ay)$ in Q where $m, n=0, 1, 2, \cdots$. Under the hypotheses of Theorem 3, $L[V]$ converges boundedly in R. Set $R_i(s, t)=a^{m-i}(-b)^{n-i}i!(m+n-i)!(as-bt)^{-m-n-1+i}$ and $\theta_i(z)=z^i/i!$ for $z>0$. Then for $ab\leqq 0$*

$$
(2.4)\qquad
\begin{aligned}
v(s, t) &= \sum_{i=0}^{n} \binom{n}{i} a^{i+1} R_i(s, t) L_t[\theta_i J_a] \\
&\quad + \sum_{i=0}^{m} \binom{m}{i} (-b)^{i+1} R_i(s, t) L_s[\theta_i J_t].
\end{aligned}
$$

For $ab>0$ formula (2.4) *holds when $as-bt\neq 0$ and*

$$
(2.5)\qquad
\begin{aligned}
v\left(\frac{b}{a}t, t\right) &= \left(\frac{a}{b}\right)^{m+1} L_t[\theta_{m+n+1} J_a] m! n! \\
&\quad \cdot \sum_{i=0}^{m} (-1)^{m+i} \binom{m+n-i}{n} \binom{m+n+1}{i}.
\end{aligned}
$$

PROOF. The region of convergence of $L[W]$ and formulas for $w(s, t)$ are given in Theorem 3. Since $V(x, y)=x^m y^n W(x, y)$, its transform converges boundedly in R to $v(s, t)=(-1)^{m+n} w_{mn}(s, t)$ by Theorem 2.1 of [1].

For $ab\leqq 0$, formula (2.4) results on performing the indicated differentiations on (2.1) and recognizing that the ith derivative of $L_t[J_a]$ is $(-1)^i i! L_t[\theta_i J_a]$.

When $ab>0$, (2.4) is established as above for any $(s, t)\in R$ such that $as-bt\neq 0$. There are always points in R at which $as-bt=0$, namely $(bt/a, t)$ with $\tau>|a|k_a$. To obtain a formula which holds at these points, multiply through (2.4) by $(as-bt)^{m+n+1}$. Since the functions in the resulting equation are holomorphic in R, we can differentiate them $m+n+1$ times with respect to s and take limits as $s\to bt/a$. The left member approaches $a^{m+n+1}(m+n+1)!w(bt/a, t)$ as a limit, using the fact that $v(s, t)$ is itself holomorphic, and the first sum on the right $\to 0$. Formula (2.5) is obtained by equating the limits of the left side and of the differentiated second sum.

3. **Integration along a line.** A familiar line integral in the plane is $\int_0^x F(u, y)du$ where the integration is taken along a line parallel to the x-axis, from the y-axis to a point P: (x, y) in $\overline{Q}$. A natural extension of this is the following: let L_1 and L_2 be two nonparallel lines, making angles of α and β respectively with the positive x-axis where $0\leqq\alpha$, $\beta<\pi$. Let P: (x, y) be any point of $\overline{Q}$ and consider a line L through P, parallel to L_1 and intersecting L_2 at a point P_2. The parametric equations of L and L_2 are

$$
L:\ \begin{aligned} X &= x - \xi\cos\alpha, \\ Y &= y - \xi\sin\alpha, \end{aligned}
\qquad
L_2:\ \begin{aligned} X &= x_0 + \eta\cos\beta, \\ Y &= y_0 + \eta\sin\beta, \end{aligned}
\qquad (-\infty<\xi, \eta<\infty)
$$

where (x_0, y_0) is a fixed point. The integral of F along L from P to P_2 is defined as

$$W(x, y) = \int_0^{\xi'} F(x - \xi \cos \alpha, y - \xi \sin \alpha) d\xi \tag{3.1}$$

where $\xi' = [(x - x_0) \sin \beta - (y - y_0) \cos \beta] \csc (\beta - \alpha)$ is the value of the parameter ξ corresponding to P_2.

The function W is defined for $(x, y) \in \overline{Q}$ but the path of integration may extend outside the first quadrant so that F may need to be defined in a region larger than $\overline{Q}$. We shall assume that F is defined everywhere in E^2 and is summable in every finite rectangle. (Set $F = 0$ where it is undefined.) In addition, let F satisfy conditions sufficient to insure the existence of W; for example, let F be continuous in E^2.

THEOREM 5. *Let $W(x, y)$ be given by* (3.1) *for $(x, y) \in \overline{Q}$ where $0 \leqq \alpha$, $\beta < \pi$, $\alpha \neq \beta$. Set $W_1(x) = W(x, 0)$ and $W_2(y) = W(0, y)$. Let F have a boundedly convergent transform over Q and let $L_s[W_1]$ and $L_t[W_2]$ converge for $\sigma > \sigma_0 \geqq 0$, $\tau > \tau_0 \geqq 0$. Then $L[W]$ converges boundedly to $w(s, t)$. For $0 \leqq \alpha \leqq \pi/2$*

$$w(s, t) = \frac{f(s, t) + w_1(s) \sin \alpha + w_2(t) \cos \alpha}{s \cos \alpha + t \sin \alpha} \qquad (\sigma > \sigma_0, \tau > \tau_0). \tag{3.2}$$

For $\pi/2 < \alpha < \pi$ formula (3.2) *holds when $s \cos \alpha + t \sin \alpha \neq 0$ and*

$$w(s, -s \cot \alpha) = f_t(s, -s \cot \alpha) \csc \alpha + w_2'(-s \cot \alpha) \cot \alpha \tag{3.3}$$

in the region $\sigma > \max [\sigma_0, \tau_0 |\tan \alpha|]$, $\tau > \max [\sigma_0 |\cot \alpha|, \tau_0]$.

PROOF. Let $\alpha \neq 0$ or $\pi/2$. For $(x, y) \in \overline{Q}$ define

$$\begin{aligned} V(x, y) &= \int_0^{x \sec \alpha} F(x - \xi \cos \alpha, y - \xi \sin \alpha) d\xi \qquad (x \tan \alpha \leqq y), \\ &= \int_0^{y \csc \alpha} F(x - \xi \cos \alpha, y - \xi \sin \alpha) d\xi \qquad (x \tan \alpha \geqq y). \end{aligned}$$

If $0 < \alpha < \pi/2$, then V is a convolution about an axis whose transform may be found by Theorem 5.3 of [1]. If $\pi/2 < \alpha < \pi$, then V is always given by the first formula and its transform is obtained from the corollary to Theorem 6.1 of [1].

Set $P(z) = W_2(z)$ for $z \geqq 0$, $P(z) = W_1(-z \cot \alpha)$ for $z < 0$, $0 < \alpha < \pi/2$, $P(z) = 0$ for $z < 0$, $\pi/2 < \alpha < \pi$. Then

$$\begin{aligned} P(y - x \tan \alpha) &= W_2(y - x \tan \alpha) \text{ for } x \tan \alpha \leqq y \\ &= W_1(x - y \cot \alpha) \text{ for } x \tan \alpha \geqq y. \end{aligned}$$

W_1 is summable in every finite interval since $\int_0^x W_1(u) du$ equals the double integral of $F(u, v)$ over a bounded region; similarly for W_2. Hence, the transform of $P(y-x\tan\alpha)$ may be obtained from Theorem 3.

It can be shown that $W(x, y)=V(x, y)+P(y-x\tan\alpha)$ in $\bar{Q}$. Combination of the above results yields a formula for $w(s, t)$ and its region of bounded convergence. In case $\pi/2<\alpha<\pi$, W_1 and W_2 are not independent; from the last equation we find

$$W(x, 0) = \int_0^{x\sec\alpha} F(x-\xi\cos\alpha, -\xi\sin\alpha)d\xi + W(0, -x\tan\alpha).$$

This equation, transformed by means of Theorem 4.3 of [1] gives a relation between the one-dimensional transforms of W_1 and W_2. Substitution in the expression for $w(s, t)$ results in the formulas of the theorem for this case.

When $\alpha=0$, (3.1) becomes $W(x, y)=\int_0^{\xi'} F(x-\xi, y)d\xi$ where $\xi'=(x-x_0)-(y-y_0)\cot\beta$. Hence, $W(x, y)=\int_0^x F(u, y)du+W(0, y)$ from which the theorem follows. The proof for $\alpha=\pi/2$ is similar.

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University of Connecticut and
Goucher College