THE RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS ON ALGEBRAS

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1. Introduction. Let $\mathfrak{A}$ be a finite dimensional associative algebra with an identity over the real or complex field $\mathfrak{F}$, and let f be a function on $\mathfrak{A}$ to $\mathfrak{A}$, i.e., a function with domain and range in $\mathfrak{A}$. $\mathfrak{A}$ is a normable ring, hence a metric topological space in the metric induced by the chosen norm [4]. Consequently, the usual elementary concepts of limits and continuity make sense, and the customary elementary theorems are valid.

In [2] a generalized difference quotient definition of differentiability and derivative was given for a function f for the case of a total matrix algebra over $\mathfrak{F}$. This definition, which is equally applicable to any finite dimensional associative algebra $\mathfrak{A}$ over $\mathfrak{F}$, is:

DEFINITION. *Let $f(\xi)$ be a function with domain and range in $\mathfrak{A}$ defined in some neighborhood of $\xi=\alpha$. Then $f(\xi)$ is said to be differentiable at $\xi=\alpha$, if, for all $\delta\in\mathfrak{A}$ in a sufficiently small neighborhood $\mathfrak{N}$ of* 0,

(I) *the difference $f(\alpha+\delta)-f(\alpha)$ is expressible as a finite sum of the form $f(\alpha+\delta)-f(\alpha)=\sum_{i=1}^{k}\lambda_i\delta\mu_i$ where $\lambda_i,\ \mu_i\in\mathfrak{A}$, and*

(II) $\lim_{\delta\to 0}\sum_{i=1}^{k}\lambda_i\mu_i$ *exists.*

If I *and* II *are fulfilled, then the limit in* II *is called the derivative of $f(\xi)$ at $\xi=\alpha$, and is denoted by $f^I(\alpha)$.*

If $\mathfrak{A}$ is commutative, it is easily verified that the above definition implies that $f(\xi)$ is Fréchet-differentiable at $\xi=\alpha$, which in turn implies that $f(\xi)$ is continuous at $\xi=\alpha$ [1]. For noncommutative $\mathfrak{A}$ this inference is not warranted.

The proofs in [2] of the uniqueness of the derivative, and of the theorems concerning differentiability and derivative of the sum or product of two functions, are equally valid for the more general algebras $\mathfrak{A}$ of the type considered here. However, the proof that the product fg of two functions is differentiable at $\xi=\alpha$ if f and g are, and that $(fg)^I(\alpha)=f^I(\alpha)g(\alpha)+f(\alpha)g^I(\alpha)$, assumed that at least one of the functions f, g was continuous at $\xi=\alpha$, and it was conjectured that (A) *this hypothesis is essential* since (B) *differentiability at a point does not imply continuity at the point.*

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In this note the above double conjecture is settled for any $\mathfrak{A}$ as defined above by disproving Conjecture A, and proving Conjecture B, together with some theorems which get at the essential reason for the validity of B and settle a question of N. J. Fine.

2. **Differentiability of the product of differentiable functions.** As a preliminary to the treatment of Conjecture A, the definition of differentiability is recast as follows.

LEMMA 2.1. *A function $f(\xi)$ defined in a neighborhood of $\xi=\alpha\in\mathfrak{A}$, is differentiable at $\xi=\alpha$ if and only if, for all $\delta\in\mathfrak{A}$ with sufficiently small norm, the functional difference can be written in the form*

$$f(\alpha+\delta)-f(\alpha)=\delta\Phi+\sum_{i=1}^{m}\nu_i\delta\eta_i, \qquad (\Phi,\nu_i,\eta_i\in\mathfrak{A})$$

where $\lim_{\delta\to 0}\Phi$ *exists and*

$$\sum_{i=1}^{m}\nu_i\eta_i=0.$$

PROOF. The sufficiency of the condition follows directly from the definition. For the necessity, we observe that from condition I of the definition, for $\|\delta\|$ sufficiently small, $f(\alpha+\delta)-f(\alpha)=\delta\sum_{i=1}^{k}\lambda_i\mu_i+[\sum_{i=1}^{k}\lambda_i\delta\mu_i-\delta\sum_{i=1}^{k}\lambda_i\mu_i]$. Since $\mathfrak{A}$ has an identity, the bracketed expression is of the form $\sum_{i=1}^{m}\nu_i\delta\eta_i$. Further $\sum_{i=1}^{m}\nu_i\eta_i=0$, and condition II of the definition assures that $\lim_{\delta\to 0}\sum_{i=1}^{k}\lambda_i\mu_i$ exists.

Conjecture A is now settled in the negative by

THEOREM 2.1. *Let $f(\xi)$ and $g(\xi)$ be functions differentiable at $\xi=\alpha\in\mathfrak{A}$. Then $p(\xi)=f(\xi)g(\xi)$ is differentiable at $\xi=\alpha$ and $p^I(\alpha)=f^I(\alpha)g(\alpha)+f(\alpha)g^I(\alpha)$.*

PROOF. By the lemma

$$f(\alpha+\delta)=f(\alpha)+\delta\Phi+\sum_{i=1}^{k}\nu_i\delta\eta_i,$$

$$g(\alpha+\delta)=g(\alpha)+\delta\Psi+\sum_{i=1}^{m}\theta_i\delta\pi_i$$

such that $\lim_{\delta\to 0}\Phi=f^I(\alpha)$, $\lim_{\delta\to 0}\Psi=g^I(\alpha)$, and

$$\sum_{i=1}^{k}\nu_i\eta_i=\sum_{i=1}^{m}\theta_i\pi_i=0 \text{ for all } \delta\in\mathfrak{A} \text{ in some neighborhood of } 0. \tag{2.1}$$

Therefore,

$$
\begin{aligned}
p(\alpha+\delta)-p(\alpha) &= f(\alpha+\delta)g(\alpha+\delta)-f(\alpha)g(\alpha) \\
&= \left[f(\alpha)+\delta\Phi+\sum_{i=1}^{k}\nu_i\delta\eta_i\right]\left[g(\alpha)+\delta\Psi+\sum_{i=1}^{m}\theta_i\delta\pi_i\right] \\
&\quad -f(\alpha)g(\alpha) \\
&= f(\alpha)\delta\Psi+f(\alpha)\sum_{i=1}^{m}\theta_i\delta\pi_i+\delta\Phi g(\alpha)+\delta\Phi\delta\Psi \\
&\quad +\delta\Phi\sum_{i=1}^{m}\theta_i\delta\pi_i+\left[\sum_{i=1}^{k}\nu_i\delta\eta_i\right]\delta\Psi \\
&\quad +\left[\sum_{i=1}^{k}\nu_i\delta\eta_i\right]g(\alpha)+\left[\sum_{i=1}^{k}\nu_i\delta\eta_i\right]\left[\sum_{i=1}^{m}\theta_i\delta\pi_i\right].
\end{aligned}
$$

Thus the difference $p(\alpha+\delta)-p(\alpha)$ satisfies condition I of the definition of differentiability. To verify condition II, we need to establish the existence of the limit of a "detached coefficient" of δ. Any such coefficient will do, by virtue of the uniqueness of the derivative, when it exists [2]. One such "detached coefficient" is,

$$
\begin{aligned}
f(\alpha)\Psi+\Phi g(\alpha)+\delta\Phi\Psi+f(\alpha)\sum_{i=1}^{m}\theta_i\pi_i+\delta\Phi\sum_{i=1}^{m}\theta_i\pi_i+\left[\sum_{i=1}^{k}\nu_i\eta_i\right]\delta\Psi \\
+\left[\sum_{i=1}^{k}\nu_i\eta_i\right]g(\alpha)+\left[\sum_{i=1}^{k}\nu_i\eta_i\right]\left[\sum_{i=1}^{m}\theta_i\delta\pi_i\right]
\end{aligned}
$$

which by equations 2.1, is

$$f(\alpha)\Psi+\Phi g(\alpha)+\delta\Phi\Psi.$$

The limit as $\delta\to 0$ of this expression exists, since the limits of Φ and Ψ exist, and that limit is $f(\alpha)g^I(\alpha)+f^I(\alpha)g(\alpha)$.

3. **Differentiability does not imply continuity.** Theorem 2.1 was proved without invoking continuity of f or g at α as in [2]. Indeed, the following example, communicated by N. J. Fine, shows that a function may be differentiable at $\alpha\in\mathfrak{A}$ and discontinuous at α.

Let $\mathfrak{A}$ be the algebra of 2×2 matrices over the real field. For

$$\xi=\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}$$

let $M_\xi=\max_{i,j}|x_{ij}|$. Define a function $f(\xi)$ by,

$$f(\xi) = \begin{pmatrix} x_{11} & M_\xi^2 x_{12} \\ x_{21}/M_\xi^2 & x_{22} \end{pmatrix}, \qquad \text{for } \xi \neq 0$$

$$f(0) = 0.$$

$f(\xi)$ is discontinuous at $\xi=0$, since $\lim_{\xi\to 0} f(\xi)$ does not exist. However,

$$f(0+\delta) - f(0) = f(\delta) = P^{-1}\delta P$$

where

$$P = \begin{pmatrix} M_\delta & 0 \\ 0 & 1/M_\delta \end{pmatrix} \qquad \text{for } \delta \neq 0,$$

$$P = I, \text{ the identity matrix}, \qquad \text{for } \delta = 0.$$

Hence f fulfills condition I for differentiability at $\xi=0$. Further $\lim_{\delta\to 0} PP^{-1} = I$. Hence $f(\xi)$ is differentiable at $\xi=0$.

The example is, so to speak, a pointwise example, and Fine, in the cited communication, raised the question: Is a function which is differentiable in a neighborhood of $\alpha \in \mathfrak{A}$, necessarily continuous at α? That this is not the case, and why it is not, is shown by the following theorem.

THEOREM 3.1. *Let $\epsilon_1, \cdots, \epsilon_n$ be a basis for $\mathfrak{A}$, with ϵ_1 the identity of $\mathfrak{A}$. With $\xi = \sum_{i=1}^n x_i\epsilon_i$, $x_i \in \mathfrak{F}$, let $f(\xi) = \sum_{i=1}^n f_i(x_1, \cdots, x_n)\epsilon_i$, $f_i(x_1, \cdots, x_n) \in \mathfrak{F}$, be defined in a neighborhood $\mathfrak{N}$ of $\alpha = \sum_{i=1}^n a_i\epsilon_i$, $a_i \in \mathfrak{F}$. A necessary condition that $f(\xi)$ be differentiable at $\xi=\alpha$ is that $\partial f_i/\partial x_1$ exist at $\xi=\alpha$ for $i=1, \cdots, n$. In this case*

$$f^I(\alpha) = \sum_{i=1}^n \left.\frac{\partial f_i}{\partial x_1}\right]_{\xi=\alpha} \epsilon_i.$$

Further, if $\mathfrak{A}$ is normal simple over $\mathfrak{F}$, then this condition is also sufficient.

PROOF. (a) *Necessity.* Since $f(\xi)$ is differentiable at $\xi=\alpha$,

$$f(\alpha+\delta) - f(\alpha) = \sum_{i=1}^k \lambda_i\delta\mu_i$$

and $\lim_{\delta\to 0} \sum_{i=1}^k \lambda_i\mu_i$ exists. Choose $\delta = d\epsilon_1$, $d \in \mathfrak{F}$. Then

$$\lim_{d\to 0} [f(\alpha + d\epsilon_1) - f(\alpha)]/d = \lim_{d\to 0} \sum_{i=1}^k \lambda_i\mu_i = f^I(\alpha)$$

exists. This implies that the limit of each basis component exists, i.e.,

$$\lim_{d\to 0}\frac{f_i(a_1+d, a_2, \cdots, a_n) - f_i(a_1, a_2, \cdots, a_n)}{d} = \frac{\partial f_i}{\partial x_1}\bigg]_{\xi=\alpha}$$

exists. Hence

$$f^I(\alpha) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_1}\bigg]_{\xi=\alpha} \epsilon_i.$$

(b) *Sufficiency, under the added hypothesis that $\mathfrak{A}$ is normal simple.*

Let $\delta = \sum_{i=1}^{n} d_i\epsilon_i$, $d_i \in \mathfrak{F}$, such that $(\alpha+\delta) \in \mathfrak{N}$. We seek to show first that the difference

$$f(\alpha+\delta) - f(\alpha) = \sum_{i=1}^{n} [f_i(\alpha+\delta) - f_i(\alpha)]\epsilon_i,$$

where $f_i(\xi)$ means $f_i(x_1, \cdots, x_n)$, can be written in the form $\sum_{i=1}^{k} \lambda_i\delta\mu_i$, or expressing δ, λ_i, μ_i in terms of the basis elements, that

$$(3.1) \qquad f(\alpha+\delta) - f(\alpha) = \sum_{i,j=1}^{n} t_{ij}\epsilon_i\left(\sum_{k=1}^{n} d_k\epsilon_k\right)\epsilon_j.$$

If c_{ijk} are the multiplication constants for the basis $\epsilon_1, \cdots, \epsilon_n$, i.e., $\epsilon_i\epsilon_j = \sum_{k=1}^{n} c_{ijk}\epsilon_k$, then 3.1 can be written

$$(3.2) \qquad f(\alpha+\delta) - f(\alpha) = \sum_{i,j,k,r,s=1}^{n} t_{ij}d_k c_{iks}c_{sjr}\epsilon_r.$$

Equation 3.2 will be fulfilled for arbitrary δ in $\mathfrak{N}$ i.e., arbitrary d_k, if and only if the system of linear equations over $\mathfrak{F}$,

$$(3.3) \qquad f_r(\alpha+\delta) - f_r(\alpha) = \sum_{s,i,j=1}^{n} t_{ij}c_{iks}c_{sjr}, \qquad k, r = 1, \cdots, n$$

is solvable for the coefficients t_{ij}. Now $f(\alpha+\delta) - f(\alpha)$ can be written in the form

$$f(\alpha+\delta) - f(\alpha) = \sum_{r=1}^{n} [f_r(\alpha+\delta) - f_r(\alpha)]\epsilon_r = \sum_{r=1}^{n}\left(\sum_{k=1}^{n} g_{rk}d_k\right)\epsilon_r,$$

where

$$g_{rs} = [f_r(a_1+d_1, \cdots, a_s+d_s, a_{s+1}, \cdots, a_n)$$
$$\qquad - f(a_1+d_1, \cdots, a_{s-1}+d_{s-1}, a_s, \cdots, a_n)]d_s^{-1}, \qquad \text{if } d_s \neq 0,$$

$$g_{rs} = 0, \qquad \text{if } s \neq 1 \text{ and } d_s = 0,$$

$$g_{r1} = \frac{\partial f_r}{\partial x_1}, \qquad \text{if } d_1 = 0.$$

Hence the problem of satisfying 3.2 for arbitrary d_k is transformed to that of showing that the system of linear equations over $\mathfrak{F}$,

$$\sum_{s,i,j} t_{ij}c_{iks}c_{sjr} = g_{rk} \qquad (r, k = 1, \cdots, n), \tag{3.4}$$

is solvable for the t_{ij}. Since $\mathfrak{A}$ is normal simple, the coefficient matrix of the t_{ij} is nonsingular [**3**], and 3.4 has a unique solution for the t_{ij}. Hence

$$\begin{aligned} f(\alpha + \delta) - f(\alpha) &= \sum_{r,k}^{n} g_{rk}d_k\epsilon_r \\ &= \sum_{i,j,k,r,s=1}^{n} t_{ij}c_{iks}c_{sjr}d_k\epsilon_r \\ &= \sum_{i,j=1}^{n} t_{ij}\epsilon_i\delta\epsilon_j \end{aligned}$$

and the function f fulfills requirement I of the definition for $\alpha+\delta$ in $\mathfrak{N}$.

Now $\lim_{\delta\to 0} \sum_{i,j=1}^{n} t_{ij}\epsilon_i\epsilon_j$ exists if and only if the limit of its rth component, $r=1, \cdots, n$, exists. The rth component is $\sum_{i,j=1}^{n} t_{ij}c_{ijr} = \sum_{i,j,k=1}^{n} t_{ij}c_{i1k}c_{kjr}$, since $c_{i1k}=1$ if $k=i$, 0 otherwise, and

$$\begin{aligned} \lim_{\delta\to 0} \sum_{i,j} t_{ij}c_{i1k}c_{kjr} &= \lim_{\delta\to 0} g_{r1} = \lim_{d_1\to 0} g_{r1} \\ &= \lim_{d_1\to 0} \frac{f_r(a_1 + d_1, a_2, \cdots, a_n) - f_r(a_1, \cdots, a_n)}{d_1} \\ &= \left.\frac{\partial f_r}{\partial x_1}\right]_{\xi=\alpha}. \end{aligned}$$

This holds for each $r=1, \cdots, n$, hence $f(\xi)$ is differentiable at $\xi=\alpha$.

Theorem 3.1 shows that for a normal simple algebra $\mathfrak{A}$, e.g., a total matrix algebra, differentiability of $f(\xi)$ at $\xi=\alpha$, requires only that the component functions of $f(\xi)$ be differentiable with respect to the identity component of ξ; they may be any functions whatever, continuous or not, of $x_2, \cdots, x_n$, provided only that $f(\xi)$ is defined in $\mathfrak{N}$.

Normal simple algebras are the most general algebras satisfying the second part of Theorem 3.1 in the following sense.

THEOREM 3.2. *If $\mathfrak{A}$ is not normal simple, then there exists at least one function on $\mathfrak{A}$ to $\mathfrak{A}$, with analytic component functions, which is not differentiable at any element of $\mathfrak{A}$.*

PROOF. Suppose $\mathfrak{A}$ is not simple. Then $\mathfrak{A}$ contains a nontrivial two-sided ideal $\mathfrak{C}$. The identity ϵ_1 of $\mathfrak{A}$ does not belong to $\mathfrak{C}$. Since $\mathfrak{C}\neq 0$,

$\exists$ an element $\epsilon_2 \in \mathfrak{C}$ independent of ϵ_1 over $\mathfrak{F}$. Then $\mathfrak{A}$ has a basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ including ϵ_1 and ϵ_2. The function $f(\xi) = x_2\epsilon_1$ has component functions analytic for all $\xi \in \mathfrak{A}$. If $f(\xi)$ were differentiable at some $\alpha \in \mathfrak{A}$, then for all δ with $\|\delta\|$ sufficiently small,

$$f(\alpha + \delta) - f(\alpha) = \sum_{i,j=1}^{n} t_{ij}\epsilon_i\delta\epsilon_j.$$

Choose $\delta = d_2\epsilon_2 \in \mathfrak{C}$ with $d_2 \neq 0$ in $\mathfrak{F}$, and $|d_2|$ appropriately small. Then for all such d_2,

$$f(\alpha + d_2\epsilon_2) - f(\alpha) = d_2\epsilon_1 = \sum_{i,j=1}^{n} t_{ij}\epsilon_i d_2\epsilon_2\epsilon_j,$$

whence

$$\epsilon_1 = \sum_{i,j=1}^{n} t_{ij}\epsilon_i\epsilon_2\epsilon_j$$

is in $\mathfrak{C}$, a contradiction. Hence $f(\xi)$ is differentiable at no $\alpha \in \mathfrak{A}$.

Suppose that $\mathfrak{A}$ is not normal. Then $\exists$ an element $\epsilon_2 \in \mathfrak{A}$, independent of ϵ_1, lying in the centrum of $\mathfrak{A}$. Consider again $f(\xi) = x_2\epsilon_1$ and suppose that $f(\xi)$ is differentiable at $\xi = \alpha$. Choosing $\delta = d_2\epsilon_2$, then as above

$$\epsilon_1 = \sum_{i,j=1}^{n} t_{ij}\epsilon_i\epsilon_2\epsilon_j = \left(\sum_{i,j=1}^{n} t_{ij}\epsilon_i\epsilon_j\right)\epsilon_2. \tag{3.5}$$

Since $\lim_{\delta\to 0} \sum_{i,j=1}^{n} t_{ij}\epsilon_i\epsilon_j$ exists, 3.5 implies that $\epsilon_1 = f^I(\alpha)\epsilon_2$. But by Theorem 3.1, $f^I(\alpha) = \partial f/\partial x_1|_{\xi=\alpha} = 0$. Hence $\epsilon_1 = 0$, a contradiction.

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