A NOTE ON PARACOMPACTNESS AND NORMALITY OF MAPPING SPACES[1]

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In a discussion with the author, J. C. Moore asked whether the space Y^X of mappings of a compact metric space X in a locally compact paracompact space Y is necessarily paracompact. (Throughout, "space" means "Hausdorff space," all mappings are understood to be continuous, and Y^X is given the compact-open topology.) It has been shown by Fadell [1] that if we drop the assumption that Y is locally compact the answer is negative. We shall give two simple examples to show that not even the assumption that Y is compact is sufficient. In both examples, as in Fadell's, X will be the unit interval I, so that Y^X becomes the space of paths on X. In our first example, Y is 1-dimensional (in the covering sense); in our second, Y has infinite dimension but Y^X is not even normal. In Fadell's example, Y (though not compact) is 1-dimensional and Y^X is also not normal. It would be interesting to know whether Y^I is normal whenever Y is both compact and finite-dimensional.

Let L be the transfinite line from 0 to ω_1 inclusive; this is the compact, totally ordered 1-dimensional space obtained by joining each countable ordinal to its successor by an interval, and adjoining ω_1 (see, for example, [2, p. 263] for a formal definition of $L-(\omega_1)$).

THEOREM 1. *L^I is not paracompact.*

Let M be the subset of L^I consisting of those maps $f: I\rightarrow L$ for which $f(0)=0$. Clearly M is closed in L^I, and it will suffice to prove M not paracompact. In what follows, α will run over the ordinals $<\omega_1$. Put $U_\alpha=\{f|f\in M$ and, for all $t\in I, f(t)<\alpha\}$; the sets U_α form an open covering of M. Let $\mathcal{V}$ be any (open) refinement of $\{U_\alpha\}$; we show that $\mathcal{V}$ is not locally finite.

We define a sequence of maps $f_n\in M$ and of countable ordinals $\alpha(n)$ $(n=1, 2, \cdots)$ as follows. Take $f_1\in M$ such that $f_1(1)>1$. Then $f_1\in$some $V_1\in\mathcal{V}$, and $V_1\subset$ some $U_{\alpha(1)}$. When f_n and $\alpha(n)$ have been defined, take $f_{n+1}\in M$ such that (i) $f_{n+1}(1)>\alpha(n)$, (ii) $f_{n+1}(t)\geqq f_n(t)$ for all $t\in I$, and (iii) $f_{n+1}(t)=f_n(t)$ whenever $0\leqq t\leqq 1-n^{-1}$. This is possible, because the subspace of L consisting of points $\leqq\alpha(n)+1$ is order-homeomorphic to I. Then take $V_{n+1}\in\mathcal{V}$ containing f_{n+1}, and

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choose $\alpha(n+1)$ so that $V_{n+1} \subset U_{\alpha(n+1)}$. Clearly $\alpha(1) < \alpha(2) < \cdots$; put $\sup \alpha(n) = \beta(<\omega_1)$. It is easy to see that, for each $t \in I$, $\lim_{n\to\infty} f_n(t)$ exists; call it $g(t)$. Thus $g(0) = 0$ and $g(1) = \beta$. Further, g is continuous, because (a) if $0 \leqq t < 1$ there is a neighborhood of t on which f_n is ultimately constant, from (iii), and (b) if $t = 1$ and m is a given positive integer, we have $f_{m+1}(s) > \alpha(m)$ whenever s is close enough to 1, and then $\alpha(n) > f_n(s) > \alpha(m)$ for all $n > m$, proving $\beta \geqq g(s) \geqq \alpha(m)$. By Dini's theorem [3, p. 239] $f_n \to g$ uniformly, and hence $f_n \to g$ in M. Thus every neighborhood of g meets V_n for infinitely many values of n. But the sets V_n are all distinct, since if $k > n$ we have $V_n \subset U_{\alpha(n)}$ and $f_k \in V_k - U_{\alpha(n)}$, by (i). Hence $\mathcal{V}$ is not locally finite, and M is not paracompact.

I do not know whether L^I is normal. However, a compact (but infinite-dimensional) space Y for which Y^I is not normal can be obtained from the following lemmas:

(1) *For any space X and family of spaces Y_λ, $(\prod Y_\lambda)^X$ and $\prod(Y_\lambda^X)$ are homeomorphic.*

This is well known, and the proof is straightforward.

(2) *If a product of nonempty spaces is normal, all but at most countably many of the factor spaces are countably compact.*

This is [4, Theorem 3, Corollary].

(3) *I^I is not countably compact.*

Again this is known; the proof consists in observing that the collection of maps f_n $(n = 1, 2, \cdots)$, where $f_n(t) = t^n$, has no cluster point (for the pointwise limit is discontinuous).

THEOREM 2. *Let Y be the product of uncountably many copies of the unit interval. Then Y^I is not normal.*

For, by (1), Y^I is homeomorphic to $\prod(I_\lambda^I)$, which by (2) and (3) is not normal.

The construction employed here raises the question: Under what conditions will Y^X be compact? The following partial answers are easily obtained.

THEOREM 3. *A necessary and sufficient condition that Y^I be compact is that Y is compact and contains no arc.*

We may identify Y with a closed subset of Y^I, namely the set of constant functions; hence, if Y^I is compact, so is Y. If Y contains an arc J, then J^I is a closed subset of Y^I, and (3) above shows that Y^I cannot be compact. Conversely if Y contains no arc, then each $f \in Y^I$ must be constant; for $f(I)$ is a Peano space (it is metrizable because, Y being Hausdorff and I compact, f is closed) and so contains an arc if it has more than one point. Thus Y^I and Y are homeomorphic, which proves the sufficiency.

It is easily seen that in Theorem 3 we may replace "compact" by "countably compact" throughout, and I (in either version) by any normal arcwise connected space of more than one point. Also the condition remains necessary if I is replaced by any connected completely regular space of more than one point.

Finally, a similar argument but with I replaced by the space K formed by a single convergent sequence (plus its limit point) proves that *the only spaces Y for which Y^X is compact for all compact metric spaces X (or for $X=K$) are the spaces with at most one point.*

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References

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