NOTE ON A SUBGROUP OF THE MODULAR GROUP[1]

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Let Γ denote the 2×2 modular group; that is, the multiplicative group of 2×2 rational integral matrices of determinant 1 in which a matrix is identified with its negative. Let $\Gamma(n)$ denote the principal congruence subgroup of Γ of level n; that is, the totality of elements a of Γ such that

$$a \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n}.$$

Let G' denote the commutator subgroup of G, G any subgroup of Γ. Finally, let Γ^n denote the fully invariant subgroup of Γ generated by the nth powers of the elements of Γ. Then it is shown in [4] that

$$(\Gamma^2)' \supset \Gamma(6)$$

and the same method can be used to show that

$$(\Gamma^3)' \supset \Gamma(6).$$

Hence

$$(\Gamma^2)' \cap (\Gamma^3)' \supset \Gamma(6) \tag{1}$$

and the question arises as to the precise relationship between $(\Gamma^2)'\cap(\Gamma^3)'$ and $\Gamma(6)$. The object of this note is to prove that these are in fact equal.

We set

$$G = (\Gamma^2)', \qquad H = (\Gamma^3)'$$

and break the proof up into several lemmas.

LEMMA 1. *The group G is a free group of rank 4 and of index 3 in Γ'. The group H is a free group of rank 5 and of index 4 in Γ'.*

PROOF. It was shown in [1] that Γ^2 is the free product of two cyclic groups of order 3, and Γ^3 the free product of three cyclic groups of order 2. Now J. Nielsen has shown [2] that the commutator subgroup of the free product of k finite cyclic groups G_i of order m_i, $1 \leqq i \leqq k$ is a free group of rank

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$$1 + m_1 m_2 \cdots m_k \left\{ -1 + \sum_{i=1}^{k} \left(1 - \frac{1}{m_i} \right) \right\}.$$

This implies that G is a free group of rank 4 and H a free group of rank 5. Since Γ' is a free group of rank 2, Schreier's formula [3]

$$R = 1 + \mu(r - 1)$$

for the rank R of a subgroup of index μ in a free group of rank r shows that G is of index 3 in Γ' and H of index 4 in Γ', completing the proof of the lemma.

LEMMA 2. *We have*

$$\Gamma' = GH.$$

PROOF. We note that the product is well-defined, since G is a normal subgroup of Γ^2, H a normal subgroup of Γ^3, and $\Gamma^2 \supset \Gamma'$, $\Gamma^3 \supset \Gamma'$ (see [1]). Hence G and H are normal subgroups of Γ'.

Consider the chains

$$\Gamma' \supset GH \supset G, \qquad \Gamma' \supset GH \supset H.$$

The first implies that $(\Gamma' : GH) | 3$, and the second that $(\Gamma' : GH) | 4$. Hence $(\Gamma' : GH) = 1$ and so $\Gamma' = GH$, completing the proof of the lemma. (This version of the proof was suggested by the referee and editor.)

LEMMA 3. *We have*

$$(\Gamma' : \Gamma(6)) = 12.$$

PROOF. It is well-known that $(\Gamma : \Gamma') = 6$ (see [1] for example) and that $(\Gamma : \Gamma(6)) = 72$. Since $\Gamma \supset \Gamma' \supset \Gamma(6)$ the result follows.

We are now in a position to prove our result.

THEOREM. *We have*

$$\Gamma(6) = G \cap H.$$

PROOF. By one of the isomorphism theorems (since G, H are normal subgroups of Γ)

$$GH/G \cong H/G \cap H.$$

By Lemma 2 this reduces to

$$\Gamma'/G \cong H/G \cap H.$$

Hence

$$(\Gamma' : G) = (H : G \cap H)$$

and by Lemma 1 it follows that

$$(H:G \cap H) = 3. \tag{2}$$

Further, we have

$$(\Gamma':G \cap H) = (\Gamma':H)(H:G \cap H)$$

and again by Lemma 1 it follows that

$$(\Gamma':G \cap H) = 4(H:G \cap H). \tag{3}$$

Together with (2), (3) implies that

$$(\Gamma':G \cap H) = 12.$$

Since $G \cap H \supset \Gamma(6)$ (formula (1)) and $(\Gamma':\Gamma(6)) = 12$ (Lemma 3) it follows that $\Gamma(6) = G \cap H$, completing the proof of the theorem.

References

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