TANGENTIAL LIMITS OF FUNCTIONS OF THE CLASS S_α

JOHN R. KINNEY[1]

1. **Introduction.** The class $S_\alpha = \{f(z) = \sum_{n=0}^{\infty} c_n z^n \mid \sum n^\alpha |c_n|^2 < \infty\}$ has been discussed by Salem and Zygmund [6; 7], Broman [1] and Carleson [2]. It has been shown [1; 2] that $f(z)$ has nontangential limits at all points of the unit circle except possibly for a set whose capacity of order $1-\alpha$ is zero. Frostman [4] shows the existence of nontangential limits for Blaschke products, except in a set whose capacity of order α is zero, if the zeros a_i, $i=1, 2, \cdots$, of the product satisfy the condition $\sum(1-|a_i|)^\alpha < \infty$. He proves a similar theorem for their derivatives. Cargo [3] shows that for $1<\gamma<1/\alpha$, the Blaschke products discussed by Frostman have limits along paths meeting the circle with order of tangency $\gamma-1$, except for sets whose capacity of order $\alpha\gamma$ is zero. He proves analogous results for the successive derivatives. His conjecture that similar results hold for S_α motivates this note.

The author is indebted to Professor Cargo for letting him see a manuscript before publication.

2. **Definitions and results.** Tangential limits are defined as follows: Let $R[\tau, \psi] = \{z \mid 1-|z| \geqq |z-e^{i\psi}|^\tau, z \neq e^{i\psi}\}$. If $z_j \to e^{i\psi}$, $z_j \in R[\tau, \psi]$, $j = 1, 2, \cdots$ implies that $\lim_{j\to\infty} f(z_j)$ exists, we say that $\tau - \lim_{z\to e^{i\psi}} f(z) = \lim_{j\to\infty} f(z_j)$.

A set D has positive capacity of order β, $\beta<1$, if there exists a positive distribution of unit mass on D, μ, for which $\sup_{z_0} \int |z-z_0|^{-\beta} d\mu(z)$ is finite. Otherwise $\operatorname{cap}_\beta D = 0$.

Following Zygmund [7, p. 139], we let

$$f_q(z) = \sum c_n (in)^{-q} z^n, \qquad f^r(z) = \sum c_n (in)^r z^n$$

be the fractional integral of order q and the fractional derivative of order r of $f(z)$.

We wish to show:

THEOREM. *Let* $f(z) = \sum c_n z^n$, $\sum n^\alpha |c_n|^2 < \infty$.

(1) *If* $0 < \gamma < \alpha$, *there exists a* $\tau - \lim_{z\to e^{i\psi}} f(z)$ *for every* $\tau < (1-\gamma)/(1-\alpha)$ *except possibly for* $e^{i\psi} \in E_\gamma^0$, *where* $\operatorname{cap}_{1-\gamma} E_\gamma^0 = 0$.

(2) *If* $r<\alpha/2$, $0<\gamma<\alpha-2r$, *there exists a* $\tau - \lim_{z\to e^{i\psi}} f^r(z)$ *for every* $\tau < (1-\gamma)/(1-\alpha+2r)$ *except possibly for* $e^{i\psi} \in E_\gamma^r$ *where* $\operatorname{cap}_{1-\gamma} E_\gamma^r = 0$.

Presented to the Society, August 27, 1962; received by the editors July 14, 1961 and, in revised form, December 14, 1961.

[1] Lincoln Laboratory, operated with support from the U. S. Army, Navy, and Air Force.

(3) *If* $q<(1-\alpha)/2$, $0<\gamma<\alpha+2q$, *there exists a* $\tau-\lim_{z\to e^{i\psi}} f_q(z)$ *for every* $\tau<(1-\gamma)/(1-\alpha-2q)$, *except possibly for* $e^{i\psi}\in E^q_\gamma$, *where* $\operatorname{cap}_{1-\gamma} E^q_\gamma=0$.

3. **Proof.** Let the principal branch of $(1-z)^{\beta-1}=\sum A_n(\beta)z^n$. It has been shown [6] that $A_n(\beta)=n^{-\beta}[1+O(1/n)]/\Gamma(1-\beta)$. Let $c^0_n(\alpha/2)=c_n/A_n(\alpha/2)+O(1/n)$, $c^r_n(\alpha/2)=c_n/A_n(\alpha/2)+O^r(1/n)$, and $c^q_n(\alpha/2)=c_n/A_n(\alpha/2)+O^q(1/n)$, and let $s_0(\theta)$, $s_r(\theta)$ and $s_q(\theta)$ be functions having $\sum c^0_n(\alpha/2)e^{in\theta}$, $\sum c^r_n(\alpha/2)e^{in\theta}$, and $\sum c^q_n(\alpha/2)e^{in\theta}$ as their Fourier series. Since $\sum n^\alpha|c_n|^2<\infty$, the $O(1/n)$, $O^r(1/n)$, $O^q(1/n)$ can be chosen so that $\sum|c^0_n(\alpha/2)|^2<\infty$, $\sum|c^r_n(\alpha/2)|^2<\infty$, $\sum|c^q_n(\alpha/2)|^2<\infty$, $|s_0(\theta)|^2$, $|s_r(\theta)|^2$, $|s_q(\theta)|^2$ are integrable, and

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-ze^{-i\theta})^{\alpha/2-1}s_0(\theta)d\theta,$$

$$f^r(z)=\frac{i^r}{2\pi}\,\frac{\Gamma(1+r-\alpha/2)}{\Gamma(1-\alpha/2)}\int_{-\pi}^{\pi}(1-ze^{-i\theta})^{\alpha/2-r-1}s_r(\theta)d\theta,$$

$$f_q(z)=\frac{i^{-q}}{2\pi}\,\frac{\Gamma(1-q-\alpha/2)}{\Gamma(1-\alpha/2)}\int_{-\pi}^{\pi}(1-ze^{-i\theta})^{\alpha/2+q-1}s_q(\theta)d\theta.$$

Let $E^t_\gamma=\left\{e^{i\psi}\,\middle|\,\int_{-\pi}^{\pi}|e^{i\psi}-e^{i\theta}|^{\gamma-1}|s_t(\theta)|^2d\theta=\infty\right\}_{t=0,r,q}$. du Plessis [5] has shown that for $|s_t(\theta)|^2$ integrable, which is true in our case, $\operatorname{cap}_{1-\gamma} E^t_\gamma=0$, for $\gamma<1$.

If $\gamma<\beta<1$, $z\in R[(1-\gamma)/(1-\beta),\psi]$, for properly chosen K, $K>0$, $|e^{i\theta}-z|\ge\min_{z\in R[(1-\gamma)/(1-\beta),\psi]}|e^{i\theta}-z|\ge K|e^{i\theta}-e^{i\psi}|^{(1-\gamma)/(1-\beta)}$, and so

$$(1)\qquad \int_{-\pi}^{\pi}|e^{i\theta}-z|^{\beta-1}|s_q(\theta)|^2d\theta\le K^{\beta-1}\int_{-\pi}^{\pi}|e^{i\theta}-e^{i\psi}|^{\gamma-1}|s_q(\theta)|^2d\theta.$$

Pick $a>0$, $\epsilon>0$. By Schwarz's inequality

$$\left|\int_{\psi-a}^{\psi+a}(1-ze^{-i\theta})^{\alpha/2+q-1}s_q(\theta)d\theta\right|^2$$
$$\le\left|\int_{\psi-a}^{\psi+a}|e^{i\theta}-z|^{\alpha+2q-1-\epsilon}|s_q(\theta)|^2d\theta\right|\cdot\left|\int_{\psi-a}^{\psi+a}|e^{i\theta}-z|^{\epsilon-1}d\theta\right|.$$

Choose $\tau=(1-\gamma)/(1+\epsilon-\alpha-2q)$, $z\in R[\tau,\psi]$ and use (1) to obtain

$$\left|\int_{\psi-a}^{\psi+a}(1-ze^{-i\theta})^{\alpha/2+q-1}s_q(\theta)d\theta\right|^2$$
$$\le K^{\alpha+2q-\epsilon-1}\left|\int_{-\pi}^{\pi}|e^{i\theta}-e^{i\psi}|^{\gamma-1}|s_q(\theta)|^2d\theta\right|\cdot\left|\int_{\psi-a}^{\psi+a}|e^{i\theta}-z|^{\epsilon-1}d\theta\right|.$$

Since the second factor on the right goes to zero with a, for $e^{i\psi}\notin E^q_\gamma$, uniformly for $z\in R[\tau, \psi]$,

$$\lim_{a\to 0}\int_{\psi-a}^{\psi+a}(1 - ze^{-i\theta})^{\alpha/2+q-1}s_q(\theta)d\theta = 0. \tag{2}$$

Since $\int_{-\pi}^{\psi-a}+\int_{\psi+a}^{\pi}(1-ze^{-i\theta})^{\alpha/2+q-1}s_q(\theta)d\theta$ is analytic at $e^{i\psi}$, (2) implies the existence at $e^{i\psi}$ of

$$\tau - \lim_{z\to e^{i\psi}}\int_{-\pi}^{\pi}(1 - ze^{-i\theta})^{\alpha/2+q-1}s_q(\theta)d\theta.$$

This proves (3). The proof for (1) and (2) is the same.

Bibliography

1. A. Broman, *On two classes of trigonometric series*, Thesis, Uppsala, 1947.

2. L. Carleson, *On a class of meromorphic functions and its associated exceptional sets*, Thesis, Uppsala, 1950.

3. G. T. Cargo, *Angular and tangential limits of Blaschke products and their successive derivatives*, Canad. J. Math. **14** (1962), 334–348.

4. O. Frostman, *Sur les produits de Blaschke*, Kungl. Fysiogr. Sällsk. i Lund Hand. N.F. **12** (1942), 169–182.

5. N. du Plessis, *A theorem about fractional integrals*, Proc. Amer. Math. Soc. **3** (1952), 892–898.

6. R. Salem and A. Zygmund, *Capacity of sets and Fourier series*, Trans. Amer. Math. Soc. **59** (1946), 23–41.

7. A. Zygmund, *Trigonometric series*, Vol. 2, 2nd ed., Cambridge Univ. Press, Cambridge, 1959.

Lincoln Laboratory