Invariant subspaces in $L^1$

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1. Denote by $L^1$ and $L^2$ the spaces of summable and square summable functions on the circle group and by $H^1$ and $H^2$ the subspaces of $L^1$ and $L^2$ consisting of those functions whose Fourier coefficients vanish for negative indices. A subspace $M$ of $L^1$ or of $L^2$ is said to be invariant if

$$x \cdot M \subseteq M$$

and to be doubly invariant if also

$$\bar{x} \cdot M \subseteq M$$

where $x$ is the character

$$x(e^{is}) = e^{is}.$$

$H^1$ and $H^2$ are closed subspaces which are invariant but not doubly invariant.

Invariant subspaces on the circle were originally studied by Beurling [1] who showed that the closed invariant subspaces contained in $H^2$ are of the form $\phi \cdot H^2$ where $\phi$ is a function in $H^2$ which has modulus one a.e. Such functions are called inner functions. Rudin and de Leeuw [3, p. 476] have shown that the closed invariant subspaces in $H^1$ have the same structure as those in $H^2$. That is to say, they are of the form $\phi \cdot H^1$ where again $\phi$ is an inner function. The arguments given in [1; 3] depend to a considerable extent on the function theory of the spaces $H^1$ and $H^2$.

Recently, Helson and Lowdenslager [2] using Hilbert space methods have given a very elegant and simple proof of Beurling's theorem.

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They establish more than Beurling did in that they show that the closed invariant subspaces of $L^2$ which are not doubly invariant are of the form $\phi \cdot H^2$ where $\phi$ is a function which has modulus one a.e. We wish to emphasize that the Helson-Lowdenslager argument is free of function-theoretic considerations.\footnote{The argument given in \cite{2} appeals to the fact that a function in $H^1$ which vanishes on a set of positive measure must vanish a.e. However it is not difficult to see that the argument goes through without using this.}

The purpose of this note is to show that a few algebraic and topological considerations together with a description of the closed invariant subspaces of $L^2$ which are not doubly invariant leads to a description of the corresponding subspaces of $L^1$. One reason for our interest in this is that from a description of these invariant subspaces one can recover a considerable amount of the function theory of $H^1$ and $H^2$.\footnote{This was first pointed out to the author by Professor Helson.}

2. Theorem 1. The closed invariant subspaces of $L^1$ which are not doubly invariant are of the form $\phi \cdot H^1$ where $\phi$ is a function which has modulus one a.e.

Proof. Let $M$ be a closed invariant subspace of $L^1$ which is not doubly invariant. Suppose for the moment that $M$ contains no doubly invariant subspaces other than the subspace consisting only of the zero vector.

Let $f \in M$ and assume $f$ is not the zero vector. We claim

$$f = \phi g$$

where $g \in H^1$, $\phi \in M$, and $\phi$ has modulus one a.e. Indeed, let

$$f = f_1 f_2$$

be any factorization of $f$ as the product of functions in $L^2$, and consider the closed invariant subspaces $M_1$ and $M_2$ in $L^2$ generated by $f_1$ and $f_2$: $M_k$ ($k = 1, 2$) is the closed linear span in $L^2$ of the functions $\chi^* f_k$ where $n = 0, 1, \ldots$. Since $M$ is invariant and closed in $L^1$,

$$M_1 \cdot M_2 \subseteq M.$$ 

In particular $f_1 \cdot M_2$ is contained in $M$, and since $M$ does not contain any nontrivial doubly invariant subspaces, $M_2$ cannot be doubly invariant. Similarly $M_1$ is not doubly invariant. Therefore by Beurling's theorem (Helson-Lowdenslager version)

$$M_k = \phi_k \cdot H^2$$

($k = 1, 2$)
where \( \phi_k \) has modulus one a.e. (1) now follows from (2), (3), and (4).

Now denote by \( J \) the set of all functions in \( M \) which have modulus one a.e. (because of (1) \( J \) is not empty), and let \( N \) be the closed invariant subspace in \( L^1 \) generated by the set \( J: N \) is the closed linear span in \( L^1 \) of the set of functions \( \chi^n\phi \) where \( \phi \in J \) and \( n \geq 0 \). Since the \( L^1 \) norm dominates the \( L^1 \) norm, \( N \) must be contained in \( M \). This in turn implies that \( N \) is not doubly invariant, and therefore

\[
N = \phi \cdot H^1
\]

where \( \phi \) has modulus one a.e.

We now claim \( M = \phi \cdot H^1 \). Since \( \phi \) belongs to \( N \), \( \phi \) also belongs to \( M \), and therefore the invariance of \( M \) implies \( \phi \cdot H^1 \subseteq M \). On the other hand if \( f \in M \), then from (1) \( f = \psi g \) where \( \psi \in J \) and \( g \in H^1 \). (5) implies \( \psi = \phi \lambda \) where \( \lambda \in H^1 \). Since \( \lambda \) must have modulus one a.e., \( \lambda g \in H^1 \), and therefore \( f \), which is equal to \( \phi \lambda g \), is in \( \phi \cdot H^1 \).

We have proved Theorem 1 under the assumption that \( M \) contains no doubly invariant subspaces other than the subspace consisting only of the zero vector. Before we remove this condition on \( M \) we need to observe that a function in \( M \) which vanishes on a set of positive measure must vanish a.e. To this end let \( f \) be a nonzero function in \( M \) and denote by \( M_f \) the closed invariant subspace in \( L^1 \) generated by \( f \). Then \( M_f \) is contained in \( M \), and since \( M \) does not contain any nontrivial doubly invariant subspaces, the same must be true of \( M_f \). Therefore \( M_f = \phi \cdot H^1 \). Since \( \phi \in M_f \), there is a sequence \( g_n \) of trigonometric polynomials such that \( g_nf \) converges to \( \phi \) in the metric of \( L^1 \). Since \( \phi \) has modulus one a.e., \( f \) cannot vanish on any set of positive measure.

In particular, since \( H^1 \) contains no nontrivial doubly invariant subspaces, a function in \( H^1 \) which vanishes on a set of positive measure must vanish a.e. It is this well known fact that we need to complete the proof of Theorem 1.

Suppose now that \( M \) is invariant and that \( M \) contains a nontrivial doubly invariant subspace \( N \). Denote by \( K \) the annihilator of \( M: K \) consists of all functions \( g \in L^\infty \) such that

\[
\int fg d\sigma = 0
\]

for all \( f \in M \). If \( f \in N \) and \( g \in K \), the double invariance of \( N \) implies

\[
\int fg d\sigma = 0 \quad (\text{for } f \in M)
\]

\[
\int fg d\sigma = (1/2\pi) \int 2\pi f(e^{i\theta})g(e^{i\theta}) d\theta
\]

The argument which follows is a variation of one which appears in [2, p. 253].

\[
\int fg d\sigma = (1/2\pi) \int 2\pi f(e^{i\theta})g(e^{i\theta}) d\theta
\]

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\[ \int x^n f g d\sigma = 0 \]

for all \( n \), and therefore \( fg \) vanishes a.e. Since \( N \) contains at least one function which does not vanish on some set of positive measure, every \( g \in K \) must vanish on a set of positive measure. If now \( f \in M \) and \( g \in K \), the invariance of \( M \) implies

\[ \int x^n f g d\sigma = 0 \]

for \( n \geq 0 \), and therefore \( fg \in H^1 \). Since \( g \) vanishes on a set of positive measure, \( fg \) must vanish a.e., and hence

\[ \int x^n f g d\sigma = 0 \]

for all \( g \in K \). This implies that \( x^n f \in M \), and \( M \) is doubly invariant.

We have shown that a closed invariant subspace of \( L^1 \) which is not doubly invariant cannot contain any nontrivial doubly invariant subspaces, which completes the proof of Theorem 1.

REFERENCES