

ON VANISHING ALGEBRAS¹

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I. Let G be a locally compact group with left invariant Haar measure m . For any measurable subset S of G , define L_S to be that subset of $L^1(G)$ consisting of all functions which vanish (a.e.) on the complement of S . When L_S forms an algebra, we call it a *vanishing algebra*. The notion and study of vanishing algebras was initiated by A. Simon. We recall here some known results (see [2]):

(1) If S is a semigroup l.a.e. (i.e., there exists a semigroup T in G such that $S=T$ locally almost everywhere), then L_S is a vanishing algebra.

(2) If S is a group l.a.e., then L_S is a self-adjoint vanishing algebra.

In this note we shall prove that the converse of (1) is true when G is unimodular and σ -compact, and the converse of (2) is always true. We shall first establish some lemmas which are essential in obtaining our results. The main theorems will appear in §§III and IV.

II. For any measurable subset S of G , let $D(S) = \{x \in G: \text{for every open neighborhood } U \text{ of } x, m(U \cap S) > 0\}$, and let $I(S) = \{x \in G: \text{there exists an open neighborhood } V \text{ of } x \text{ such that } m(V - S) = 0\}$. It is known that $D(S)$ is closed and $S - D(S)$ is l.n. (locally negligible). From the definitions we see immediately that $I(S)$ is open and $I(S) \subset D(S)$. We remark that $I(S) - S$ is l.n. For otherwise, there would exist a compact set $K \subset I(S) - S$ with $m(K) > 0$; if we cover K by finitely many of the V 's, we would have $m(K) = 0$. This is a contradiction. We remark also that if S is contained in a σ -compact subset of G , then $m(S - D(S)) = 0$ and $m(I(S) - S) = 0$.

LEMMA 1. *If L_S is a vanishing algebra, then $D(S)I(S) \cup I(S)D(S) \subset I(S)$ and $I(S)$ is a semigroup.*

PROOF. Let $p \in D(S)$ and $q \in I(S)$. Let P be an open neighborhood of p with finite measure, and let Q be an open neighborhood of q with finite measure and $m(Q - S) = 0$. Consider the convolution of the characteristic functions of $P \cap S$ and $Q \cap S$; we have $\phi_{P \cap S} * \phi_{Q \cap S}(pq) = m((pq)^{-1}(P \cap S) \cap (Q \cap S)^{-1}) = m(P \cap pqQ^{-1} \cap S) > 0$, since $P \cap pqQ^{-1}$ is an open neighborhood of p . Since this function is con-

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tinuous and is in L_S , there exists an open neighborhood N of pq such that $m(N - S) = 0$, i.e., $pq \in I(S)$. By considering $\phi_{q \cap S} * \phi_{p \cap S}(qp)$, we get similarly $qp \in I(S)$. Thus $D(S)I(S) \cup I(S)D(S) \subset I(S)$. Clearly then $I(S)$ is a semigroup.

LEMMA 2. *Let S be a measurable set which is contained in a σ -compact subset of G . If $S \subset D(S)$ and $SS \subset I(S)$, then S is a semigroup a.e.*

PROOF. Since $m(I(S) - S) = 0$, from $SS \subset I(S)$ we get $m(SS - S) = 0$. Hence L_S is a vanishing algebra. It then follows from the assumptions and Lemma 1 that $S^n \subset I(S)$ for every $n \geq 2$. Hence $S = S^\infty$ a.e., where $S^\infty = \cup S^n$ is a semigroup.

We remark that in this lemma the condition $S \subset D(S)$ can be omitted since we can replace S by $S \cap D(S)$ in the proof. It is not known, however, whether the condition $SS \subset I(S)$ can be replaced by $SS \subset I(S)$ a.e.

LEMMA 3. *Let L_S be a vanishing algebra. If $S \subset D(S)$ and $SS \subset I(S)$, then S is a semigroup l.a.e.*

PROOF. Since $I(S) - S$ is l.n., $SS - S$ is l.n. By Lemma 1 we have $S^n - S$ is l.n. for every $n \geq 2$. Thus $S = S^\infty$ l.a.e.

Again as in Lemma 2, the condition $S \subset D(S)$ can be omitted.

III. We now prove

THEOREM 1. *Suppose G is unimodular. If L_S is a vanishing algebra and S is contained in a σ -compact subset of G , then S is a semigroup a.e.*

PROOF. Let $S \subset H = \cup C_i$, where (C_i) is an increasing sequence of compact subsets of G . By the density theorem for topological groups (see [1, p. 268]) and the assumption that G is unimodular, we have that for each $S \cap C_i$, there exist bounded neighborhoods U_{ij} ($j = 1, 2, \dots$) of the identity e such that for every neighborhood U of e contained in U_{ij} ,

$$\int \left| \frac{m(S \cap C_i \cap Ux)}{m(U)} - \phi_{S \cap C_i}(x) \right| dm(x) < \frac{1}{j}.$$

This implies that for every sequence $(W_{ij}: j = 1, 2, \dots)$ of neighborhoods of e with $W_{ij} \subset U_{ij}$ for every j ,

(i) $m(S \cap C_i \cap W_{ij})/m(W_{ij})$ converges in measure to 1 on $S \cap C_i$ as $j \rightarrow \infty$.

Since G is unimodular, we have similarly a sequence $(V_{ij}: j = 1, 2, \dots)$ of bounded neighborhoods of e , such that if $W_{ij} \subset V_{ij}$ for every j ,

(ii) $m(S \cap C_i \cap xW_{ij})/m(W_{ij})$ converges in measure to 1 on $S \cap C_i$ as $j \rightarrow \infty$.

Take now symmetric $W_{ij} \subset U_{ij} \cap V_{ij}$. There is a subsequence of (W_{ij}) , which may also be denoted by (W_{ij}) , such that the expressions in (i) converges to 1 a.e. on $S \cap C_i$ and (ii) remains true. We can then choose a subsequence again to get

$$\frac{m(S \cap C_i \cap W_{ij}x)}{m(W_{ij})} \rightarrow 1 \text{ a.e. on } S \cap C_i \text{ as } j \rightarrow \infty,$$

and

$$\frac{m(S \cap C_i \cap xW_{ij})}{m(W_{ij})} \rightarrow 1 \text{ a.e. on } S \cap C_i \text{ as } j \rightarrow \infty.$$

Thus, there is a measurable set $T \subset S$ such that $m(S - T) = 0$ and $m(T \cap C_i \cap W_{ij}x)/m(W_{ij})$ and $m(T \cap C_i \cap xW_{ij})/m(W_{ij})$ both converge to 1 on $T \cap C_i$ as $j \rightarrow \infty$.

Now for any two points x and y in T , there is C_i with $x, y \in C_i \cap T$, and there exists a j such that $m(T \cap C_i \cap W_{ij}y)/m(W_{ij})$ and $m(T \cap C_i \cap xW_{ij})/m(W_{ij})$ are both $> \frac{1}{2}$. From this we get $m(W_{ij}) \geq m(W_{ij} \cap (x^{-1}T \cup yT^{-1})) = m(W_{ij} \cap x^{-1}T) + m(W_{ij} \cap yT^{-1}) - m(W_{ij} \cap x^{-1}T \cap yT^{-1}) = m(xW_{ij} \cap T) + m(W_{ij}y \cap T) - m(W_{ij} \cap x^{-1}T \cap yT^{-1}) > \frac{1}{2}m(W_{ij}) + \frac{1}{2}m(W_{ij}) - m(W_{ij} \cap x^{-1}T \cap yT^{-1})$. Therefore $m(W_{ij} \cap x^{-1}T \cap yT^{-1}) > 0$. But $m(W_{ij} \cap x^{-1}T \cap yT^{-1})$ is nothing but $\phi_{xW_{ij} \cap T} * \phi_{W_{ij}y \cap T}(xy)$. We therefore have, since L_T is a vanishing algebra, $xy \in I(T)$. Thus $TT \subset I(T)$. By Lemma 2, T is a semigroup a.e., and hence S is a semigroup a.e.

Since every compact group is unimodular, we have the following corollary which was pointed out by A. Simon [2].

COROLLARY 1. *Suppose G is compact. Then, if L_S is a vanishing algebra, S is a semigroup a.e.*

Another immediate consequence of Theorem 1 is

COROLLARY 2. *Suppose G is abelian and generated by some compact neighborhood of the identity element of G . Then, if L_S is a vanishing algebra, S is a semigroup a.e.*

The proof of Theorem 1 also suggests the following more general assertion for the nonunimodular case.

THEOREM 2. *Let L_S be a vanishing algebra. Suppose there exists a directed set $\{U_i, i \in I\}$ of symmetric neighborhoods of e with finite measures, having the property that for almost all the points x of S there exists a*

$j_x \in I$ such that $m(S \cap xU_i)$ and $m(x^{-1}U_i \cap S^{-1})$ are both $> \frac{1}{2}m(U_i)$ as $i \geq j_x$. Then S is a semigroup l.a.e. If, in addition, S is contained in a σ -compact subset of G , then S is a semigroup a.e.

PROOF. By the proof of Theorem 1 we know that there exists a measurable subset $T \subset S$, such that $m(S - T) = 0$ and $TT \subset I(T)$. The conclusions then follow from Lemmas 3 and 2.

IV. We now consider the case where L_S is a self-adjoint vanishing algebra.

THEOREM 3. *If L_S is a self-adjoint vanishing algebra, then S is a group l.a.e. If, in addition, S is contained in a σ -compact subset of G , then S is a group a.e.*

PROOF. The only nontrivial case is S not l.n. Then $D(S)$ is a group and $S = S^{-1}$ l. a.e. (see [2]). Let V be a symmetric open neighborhood of e with finite measure. We have $\phi_{V \cap S} * \phi_{V \cap S}(e) = m(V \cap S \cap (V \cap S)^{-1}) = m(V \cap S) > 0$; which implies $e \in I(S)$. Hence by Lemma 1, $D(S) \subset I(S)$ and therefore $D(S) = I(S)$. It is then clear that $S = D(S)$ l.a.e. and if S is contained in a σ -compact subset of G , $S = D(S)$ a.e. Since $D(S)$ is a group, the proof is complete.

Another interesting consequence of Lemma 1 is

THEOREM 4. *Let L_S be a vanishing algebra. If S is open, then S is a semigroup l.a.e. If, in addition, S is contained in a σ -compact subset of G , then S is a semigroup a.e.*

PROOF. By Lemma 1, $I(S)$ is a semigroup. Since S is open, $S \subset I(S)$. Therefore $S = I(S)$ l.a.e. If, in addition, S is contained in a σ -compact subset of G , we have $m(I(S) - S) = 0$, hence $S = I(S)$ a.e.

We conclude this note with one more theorem which has an interesting application.

THEOREM 5. *If L_S is a maximal vanishing algebra, then S is a closed semigroup l.a.e. If, in addition, S is contained in a σ -compact subset of G , then S is a closed semigroup a.e.*

PROOF. We observe that since L_S is a maximal vanishing algebra, $L_S = L_{D(S)}$ or $D(S) = G$. Since $D(S)$ is a closed semigroup, the proof will be completed if $D(S) \neq G$. Just suppose $D(S) = G$. Let $K \subset S$ be of positive finite measure. Then $U = \{x \in G: \phi_K * \phi_K(x) > 0\}$ is a non-empty open set (see [4]). Hence $m(U - S) = 0$. Let V be an open set contained in U such that $0 < m(V) < \infty$. Then for any $x \in G$, $\phi_{xV^{-1} \cap S} * \phi_{V \cap S}(x) = m(x^{-1}xV^{-1} \cap x^{-1}S \cap V^{-1} \cap S^{-1}) = m(V^{-1} \cap x^{-1}S) = m(xV^{-1} \cap S) > 0$. This implies $x \in I(S)$. Therefore $I(S) = G$. But

then $L_S = L_{I(S)} = L^1(G)$, which contradicts the maximality of L_S . Hence $D(S) \neq G$.

COROLLARY 3. *Let G be abelian and generated by some compact neighborhood of the identity element of G . If there exists a vanishing algebra L_S which is a maximal subalgebra in $L^1(G)$, then G is either the additive group of real numbers or the discrete integer group.*

PROOF. By Theorem 5, we may assume S to be a semigroup. The conclusion is then exactly what A. Simon [3] asserted.

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