

ON A PROBLEM OF C. BERGE

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The unsolved problem, number 4 on page 251 in [1] states: "Does the sum of two graphs have a kernel (French: noyau) if each of them has a kernel?." The purpose of this note is to give a negative answer. The definitions and notations used here are the same as in [1].

Let $G_1 = (X_1, \Gamma_1)$ where $X_1 = \{x_1, x_2, x_3, x_4\}$, and $\Gamma_1 x_1 = \{x_2, x_4\}$, $\Gamma_1 x_2 = \{x_3, x_4\}$, $\Gamma_1 x_3 = \{x_1, x_4\}$ and $\Gamma_1 x_4 = \emptyset$. Also let $G_2 = (X_2, \Gamma_2)$ where $X_2 = \{y_1, y_2\}$ and $\Gamma_2 y_1 = \{y_2\}$ and $\Gamma_2 y_2 = \emptyset$. Clearly, G_1 has a kernel, namely $\{x_4\}$, and G_2 has $\{y_2\}$ as its kernel. Form $G = G_1 + G_2 = (X_1 \times X_2, \Gamma)$. We claim that G does not have a kernel. Suppose G had one, denoted by S , then (x_4, y_2) must belong to S , because $\Gamma(x_4, y_2) = \emptyset$. By definition of S , none of the nodes in $\Gamma^{-1}(x_4, y_2) = \{(x_1, y_2), (x_2, y_2), (x_3, y_2), (x_4, y_2)\}$ can be in S . The rest of nodes of $X_1 \times X_2$, (x_1, y_1) , (x_2, y_1) and (x_3, y_1) , generate a complete subgraph (it is also an odd directed cycle), only one of them can be in S . But, no matter which one of them is in S , there is always another one, (x, y) , of them which has the property $\Gamma(x, y) \cap S = \emptyset$ where $(x, y) \notin S$. This is a contradiction to the definition of S . Hence, G goes not have a kernel.

Similarly, one can construct a family of such graphs: Take G'_1 to be a complete directed graph of n nodes ($n \geq 3$) with a Hamiltonian cycle (or take G'_1 to be a directed cycle of n nodes where n is odd and > 1), and take G_1 to be $G'_1 \cup \{x_{n+1}\}$ such that from every node of G'_1 there is a directed edge toward the node x_{n+1} and no edge leads from x_{n+1} . Take G_2 as before. Then each of G_1 and G_2 has a kernel, but $G = G_1 + G_2$ does not have one by the similar argument as before.

REFERENCE

1. C. Berge, *Théorie des graphes et ses applications*, Dunod, Paris, 1958.

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