ON A PROBLEM OF C. BERGE

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The unsolved problem, number 4 on page 251 in [1] states: "Does the sum of two graphs have a kernel (French: noyau) if each of them has a kernel?." The purpose of this note is to give a negative answer. The definitions and notations used here are the same as in [1].

Let $G_1=(X_1, \Gamma_1)$ where $X_1=\{x_1, x_2, x_3, x_4\}$, and $\Gamma_1 x_1=\{x_2, x_4\}$, $\Gamma_1 x_2=\{x_3, x_4\}$, $\Gamma_1 x_3=\{x_1, x_4\}$ and $\Gamma_1 x_4=\emptyset$. Also let $G_2=(X_2, \Gamma_2)$ where $X_2=\{y_1, y_2\}$ and $\Gamma_2 y_1=\{y_2\}$ and $\Gamma_2 y_2=\emptyset$. Clearly, G_1 has a kernel, namely $\{x_4\}$, and G_2 has $\{y_2\}$ as its kernel. Form $G=G_1+G_2=(X_1\times X_2, \Gamma)$. We claim that G does not have a kernel. Suppose G had one, denoted by S, then (x_4, y_2) must belong to S, because $\Gamma(x_4, y_2)=\emptyset$. By definition of S, none of the nodes in $\Gamma^{-1}(x_4, y_2)=\{(x_1, y_2), (x_2, y_2), (x_3, y_2), (x_4, y_2)\}$ can be in S. The rest of nodes of $X_1\times X_2$, (x_1, y_1), (x_2, y_1) and (x_3, y_1), generate a complete subgraph (it is also an odd directed cycle), only one of them can be in S. But, no matter which one of them is in S, there is always another one, (x, y), of them which has the property $\Gamma(x, y)\cap S=\emptyset$ where $(x, y)\notin S$. This is a contradiction to the definition of S. Hence, G goes not have a kernel.

Similarly, one can construct a family of such graphs: Take G_1' to be a complete directed graph of n nodes ($n\geqq 3$) with a Hamiltonian cycle (or take G_1' to be a directed cycle of n nodes where n is odd and >1), and take G_1 to be $G_1'\cup\{x_{n+1}\}$ such that from every node of G_1' there is a directed edge toward the node x_{n+1} and no edge leads from x_{n+1}. Take G_2 as before. Then each of G_1 and G_2 has a kernel, but $G=G_1+G_2$ does not have one by the similar argument as before.

Reference

1. C. Berge, *Théorie des graphes et ses applications*, Dunod, Paris, 1958.

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