

THEOREM 4.2. *A complete regular ring S is continuous if and only if $\bar{S}_l = \bar{S}_r$.*

By [3, Theorem 3] every continuous regular ring is (both left and right) self injective if it contains no nonzero strongly regular ideals. Hence we have

THEOREM 4.3. *Let S be a complete regular ring, and suppose that S does not contain any nonzero strongly regular ideals. Then $\bar{S}_l = \bar{S}_r$ (if and only if S is self injective, that is, $S = \bar{S}_l = \bar{S}_r$).*

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INVERSE LIMITS OF SOLVABLE GROUPS

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In this paper we generalize to groups of Galois type some results of P. Hall on finite solvable groups [1; 2; 3]. We need, in a modified form, some results of van Dantzig: the definition of supernatural numbers (which are related to van Dantzig's universal numbers) and Theorem 5, which he proved for ordinary p -Sylow subgroups [6]. Lemmas 1 and 4 and the method of proof in Theorem 5 are due to Tate [5].

A topological group G is of *Galois type* if it is compact and totally disconnected. In any Galois type group the open normal subgroups form a neighborhood base at the identity. Every closed subgroup is the intersection of the open subgroups containing it [4]. Whenever M and N are open normal subgroups of G and $N \supset M$ we shall write ϕ_N^M for the natural homomorphism of G/M onto G/N (these quotient groups are finite) and ϕ_N for the natural homomorphism of G onto G/N . G is the inverse limit of the groups $\{G/N\}$, N ranging over the open normal subgroups of G . Conversely, the inverse limit of finite groups is of Galois type.

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A *supernatural number* is a formal product of powers of primes. Infinitely many factors are permitted, with exponents $1, 2, \dots, \infty$. Supernatural numbers are multiplied by addition of exponents. If π is a set of primes and δ a supernatural number we say

$$\begin{aligned} \delta \text{ and } \pi \text{ are coprime} & \quad \text{when } p \in \pi \text{ implies } p \nmid \delta, \\ \pi \mid \delta \text{ (}\pi \text{ divides } \delta\text{)} & \quad \text{when } p \in \pi \text{ implies } p \mid \delta, \\ \delta \mid \pi & \quad \text{when } p \mid \delta \text{ implies } p \in \pi. \end{aligned}$$

Whenever G is a group of Galois type and S is a closed subgroup of G we define the index of S in G ($[G:S]$) as the supernatural number which is the least common multiple of the integers $[G:U]$ as U ranges over open subgroups of G containing S . Then $[G:S]$ is also the l.c.m. $\{[G:SN]\}$ as N ranges over the open normal subgroups of G . The order of $G, o(G)$, is $[G:1]$. G is a π group if $o(G) \mid \pi$. A closed subgroup S of G is a π -Sylow subgroup if π and $[G:S]$ are coprime and S is a π group.

1. LEMMA. *If S and T are closed subgroups of the Galois type group G and S contains T then $[G:T] = [G:S][S:T]$.*

PROOF. For N an open normal subgroup of G we know $[G:TN] = [G:SN][SN:TN]$ and $[SN:TN] = [S:T(S \cap N)]$. Taking the least common multiples as N varies we see that $[G:T]$ divides $[G:S][S:T]$, so we need only show that $[G:S][S:T]$ divides $[G:T]$. Suppose that for m and $k < \infty$, p^m divides $[G:S]$ and p^k divides $[S:T]$. It follows that there are open normal subgroups N_1 and N_2 of G for which $p^m \mid [G:SN_1]$ and $p^k \mid [S:T(S \cap N_2)]$. Let $N = N_1 \cap N_2$. Then $p^m \mid [G:SN]$ and $p^k \mid [S:T(S \cap N)]$. Therefore p^{m+k} divides $[G:TN]$ and thus also divides $[G:T]$. Therefore $[G:S][S:T]$ divides $[G:T]$. q.e.d.

2. COROLLARY. *A subgroup S is a π -Sylow subgroup for some π if and only if its order and index are coprime.*

3. COROLLARY. *One π -Sylow subgroup cannot strictly contain another.*

PROOF. Let S and T be π -Sylow subgroups for which $T \subset S$. Then $o(S) = o(T)[S:T]$ implies $[S:T]$ divides π , but $[G:T] = [G:S][S:T]$ implies $[S:T]$ and π are coprime, which is a contradiction unless $S = T$.

4. LEMMA. *If S is the intersection of a decreasing family $\{S_\alpha\}$ of closed subgroups of G then $[G:S] = \text{l.c.m. } \{[G:S_\alpha]\}$.*

PROOF. Lemma 1 implies $[G:S] = [G:S_\alpha][S_\alpha:S]$ for all α , hence

l.c.m. $\{[G:S_\alpha]\}$ divides $[G:S]$. Conversely, if $m < \infty$ and $p^m \mid [G:S]$ then there is an open normal subgroup N for which $p^m \mid [G:SN]$. Since $G-SN$ is compact and disjoint from $S = \bigcap S_\alpha$ there is a finite intersection $S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}$ contained in SN . However $S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}$ equals S_{α_k} since the S_α are decreasing. Therefore $p^m \mid [G:S_{\alpha_k}]$, hence $p^m \mid \text{l.c.m.}\{[G:S_\alpha]\}$. Thus $[G:S] = \text{l.c.m.}\{[G:S_\alpha]\}$. q.e.d.

A group of Galois type is *projectively solvable* if it is the inverse limit of finite solvable groups.

5. THEOREM. *Let G be projectively solvable. For each π dividing $o(G)$, G has π -Sylow subgroups, any two are conjugate, and any π subgroup of G is contained in one of them.*

PROOF. Let \mathfrak{M} be the set of closed subgroups S of G for which $[G:S]$ and π are coprime. $G \in \mathfrak{M}$ and \mathfrak{M} is partially ordered by inclusion. Lemma 4 shows that the intersection S of a maximal chain from \mathfrak{M} is minimal in \mathfrak{M} . We shall show such an S is necessarily a π group.

Let N be an open normal subgroup of G . We must show $o(S/S \cap N)$ divides π . Since G/N is solvable so is $S/S \cap N$, which is isomorphic to SN/N . Choose a π -Sylow subgroup W of $S/S \cap N$ (Hall, [1]). Let T be the inverse image in S of W under $\phi_{S \cap N}$. T is closed in G and both open and closed in S . $[G:T] = [G:S][S:T]$. $[G:S]$ and π are coprime since $S \in \mathfrak{M}$. Since π and $[S/S \cap N:W]$, which equals $[S:T]$, are coprime, we see that $[G:T]$ and π are coprime. Therefore $T \in \mathfrak{M}$. Since $T \subset S$ and S is minimal in \mathfrak{M} , $T = S$. Therefore $S/S \cap N = T/S \cap N = W$, but we know $o(W)$ divides π . Therefore S is a π -Sylow subgroup of G .

Now let T be any π subgroup of G . For each open normal subgroup N of G the set K_N of members of G whose images σ under ϕ_N satisfy $\sigma^{-1}\phi_N(T)\sigma \subset \phi_N(S)$ is closed and nonempty (Hall, [1]). $N \subset M$ implies $K_N \subset K_M$, hence $\{K_N\}$ has the finite intersection property. If $\tau \in \bigcap K_N$ then $\tau^{-1}T\tau \subset S$. Therefore any π group has a conjugate inside a given π -Sylow subgroup. Corollary 3 then implies any two π -Sylow subgroups are conjugate. q.e.d.

6. COROLLARY. *Let $\mathfrak{N}(S)$ be the normalizer of the π -Sylow subgroup S . Then $[G:\mathfrak{N}(S)]$ and π are coprime since $S \subset \mathfrak{N}(S)$.*

Theorem 5 guarantees the existence of π -Sylow subgroups. Zorn's lemma was required in the proof. Once this existence has been established, however, the π -Sylow subgroups can be characterized somewhat more constructively.

7. THEOREM. *Each π -Sylow subgroup S of G may be written as the*

inverse limit of groups $\{S_N\}$ where S_N is a π -Sylow subgroup of G/N and ϕ_N^M maps S_M onto S_N . Conversely, if $\{S_N\}$ is a family of subgroups of the G/N with these properties then the inverse limit S of $\{S_N\}$ is a π -Sylow subgroup of G .

PROOF. Any closed subgroup S of G may be written as the inverse limit of $\{S_N\}$ where $S_N = SN/N$, which is isomorphic to $S/S \cap N$. Then ϕ_N^M maps S_M onto S_N . The argument in Theorem 5 shows that S_N is a π -Sylow subgroup of G/N if S is a π -Sylow subgroup of G .

Conversely, if each S_N is a π -Sylow subgroup then S is a π group, hence S is contained in some π -Sylow subgroup S' . Then $S'N/N$ must equal S_N/N for all N , hence $S = S'$. q.e.d.

We can now generalize Hall's results on Sylow systems [3]. Let G be projectively solvable, π_G the set of primes dividing $o(G)$. Choose a $\pi_G - \{p\}$ -Sylow subgroup (p -complement) C_p for each $p \in \pi_G$. For $\Sigma \subset \pi_G$ let $S_\Sigma = \bigcap_{p \in \Sigma} C_p$. We set $S_{\pi_G} = G$. Then $\mathfrak{s} = \{S_\Sigma\}$ is a Sylow system of G .

8. THEOREM. S_Σ is a Σ -Sylow subgroup of G .

PROOF. Each C_p is the inverse limit of p -complements of the groups G/N (Theorem 7). Hall's theorem [3] tells us that the intersections of the p -complements for $p \in \Sigma$ are Σ -Sylow subgroups of the groups G/N . These Σ -Sylow subgroups are mapped onto each other by the ϕ_N^M , hence their inverse limit is a Σ -Sylow subgroup which is easily proved equal to S_Σ . q.e.d.

9. LEMMA. If $\{\Sigma_\alpha\}$ is a family of subsets of π_G then $S_{\cap \Sigma_\alpha} = \bigcap S_{\Sigma_\alpha}$.

This follows trivially from the definition of S_Σ .

10. THEOREM. $S_\Sigma S_\Delta = S_{\Sigma \cup \Delta}$.

PROOF. If $\sigma\delta \in S_\Sigma S_\Delta$ each of its projections $\phi_N(\sigma\delta)$ lies in the product of a Σ -Sylow subgroup and a Δ -Sylow subgroup of G/N (Theorem 7). By Hall's theorem [3] this product is a $\Sigma \cup \Delta$ -Sylow subgroup of G/N , and is in fact $\phi_N(S_{\Sigma \cup \Delta})$. Hence $S_\Sigma S_\Delta \subset S_{\Sigma \cup \Delta}$.

To prove the reverse inclusion, suppose $\tau \in S_{\Sigma \cup \Delta}$. Then for each open normal subgroup N , $\phi_N(\tau) \in \phi_N(S_{\Sigma \cup \Delta})$, which equals $\phi_N(S_\Sigma)\phi_N(S_\Delta)$. Thus there are elements $\sigma_N \in S_\Sigma$ and $\delta_N \in S_\Delta$ for which $\phi_N(\sigma_N\delta_N) = \phi_N(\tau)$. This shows that the set K_N of those members η of $S_\Sigma S_\Delta$ for which $\phi_N(\eta) = \phi_N(\tau)$ is nonempty. K_N is closed since $S_\Sigma S_\Delta$ is closed, even though we do not yet know it is a subgroup. $\{K_N\}$ has the finite intersection property since for open normal subgroups $N_1, \dots, N_j, \bigcap K_{N_i} \supset K_{\bigcap N_i}$. Choose $\rho \in \bigcap K_N$. For all N , $\phi_N(\rho) = \phi_N(\tau)$ implies $\rho = \tau$, but we know $\rho \in S_\Sigma S_\Delta$. Therefore $S_{\Sigma \cup \Delta} \subset S_\Sigma S_\Delta$. q.e.d.

11. COROLLARY. $S_{\Sigma}S_{\Delta} = S_{\Sigma \cup \Delta} = S_{\Delta}S_{\Sigma}$, i.e., \mathfrak{S} consists of commuting subgroups. In particular, $\{S_{\{p\}}\}$ is a commuting family of ordinary p -Sylow subgroups.

12. COROLLARY. Theorem 10 is true for infinite products, i.e., $[S_{\Sigma_{\alpha}}S_{\Sigma_{\beta}} \cdots] = S_{\cup \Sigma_{\alpha}}$ for any family $\{\Sigma_{\alpha}\}$ of subsets of π_G .

PROOF. By $S_{\Sigma_{\alpha}}S_{\Sigma_{\beta}} \cdots$ we mean the set of products of finitely many members of $\cup S_{\Sigma_{\alpha}}$. Since the $S_{\Sigma_{\alpha}}$ commute pairwise, this set, and hence its closure, S , is a subgroup. S is the smallest closed subgroup containing $S_{\Sigma_{\alpha}}$ for all α . Since $S_{\cup \Sigma_{\alpha}} \supset S_{\Sigma_{\alpha}}$ for all α we have $S_{\cup \Sigma_{\alpha}} \supset S$. It will suffice to prove that S is a $\cup \Sigma_{\alpha}$ -Sylow subgroup, for then we may apply Theorem 8 and Corollary 3. S is a $\cup \Sigma_{\alpha}$ group since it is contained in one. Since $[G:S_{\Sigma_{\alpha}}]$ and Σ_{α} are coprime and $[G:S_{\Sigma_{\alpha}}] = [G:S][S:S_{\Sigma_{\alpha}}]$ we see that for each α , $[G:S]$ and Σ_{α} are coprime. Thus $[G:S]$ and $\cup \Sigma_{\alpha}$ are coprime and S is a $\cup \Sigma_{\alpha}$ -Sylow subgroup. q.e.d.

By combining the results 8 through 12 we prove

13. THEOREM. For each projectively solvable group G there is a lattice injection of the set of subsets of π_G into the lattice of closed subgroups of G . Its range is commutative and consists of one Sylow subgroup of each type. The lattice join corresponds to set multiplication.

We now study the family of Sylow systems. The Sylow system $\tau^{-1}\mathfrak{S}\tau$ generated by the τ -conjugates of the p -complements in \mathfrak{S} is said to be conjugate to \mathfrak{S} . If $\mathfrak{S}^* = \tau^{-1}\mathfrak{S}\tau$ then $S_{\Sigma}^* = \tau^{-1}S_{\Sigma}\tau$ for all $\Sigma \subset \pi_G$.

14. CONJUGACY THEOREM. Any two Sylow systems $\mathfrak{S}, \mathfrak{S}^*$ are conjugate.

PROOF. For any finite set $\{p_1, \dots, p_n\} \subset \pi_G$ there is a $\tau \in G$ for which $\tau^{-1}C_{p_i}\tau = C_{p_i}^*$, $i \leq n$. We prove this by induction. The proof is the same as Hall's [3]. For $n = 1$ the statement is Theorem 5. Assume that conjugation with τ takes C_{p_i} into $C_{p_i}^*$, $i \leq n-1$. Since $S_{p_n}^*C_{p_n}^* = G = C_{p_n}^*S_{p_n}^*$ any conjugate of $C_{p_n}^*$ may be obtained by using an element of $S_{p_n}^*$. In particular, there is a $\sigma \in S_{p_n}^*$ for which $\sigma^{-1}(\tau^{-1}C_{p_n}\tau)\sigma = C_{p_n}^*$. Since $S_{p_n}^* \subset C_p^*$ when $p \neq p_n$ we can conclude that $(\tau\sigma)^{-1}C_{p_i}(\tau\sigma) = C_{p_i}^*$ for $i \leq n$.

Let K_p be the coset of the normalizer of C_p which takes C_p to C_p^* . K_p is closed. The induction above shows $\{K_p\}$ has the finite intersection property. Any $\tau \in \cap K_p$ clearly satisfies our requirements. q.e.d.

15. EMBEDDING THEOREM. If \mathfrak{J} is a commuting set of subgroups of G such that each $T \in \mathfrak{J}$ is a π_T -Sylow subgroup then there is a Sylow system \mathfrak{S} containing \mathfrak{J} .

PROOF. For each $p \in \pi_G$ let T_p be the closure of the product of those $T \in \mathfrak{J}$ for which $p \nmid o(T)$. (If there are no such T let $T_p = \{e\}$.) The argument in Corollary 12 shows that T_p is a group. T_p is the inverse limit of the product in each G/N of the groups $\phi_N(T)$ for which $p \nmid o(T)$, therefore $p \nmid o(T_p)$. Using Theorem 5, choose a p -complement C_p containing T_p .

Let \mathfrak{s} be the Sylow system generated by $\{C_p\}$. For each $T \in \mathfrak{J}$, S_{π_T} contains T since S_{π_T} is the intersection of the C_p for which $p \notin \pi_T$, i.e., for which $T \subset T_p \subset C_p$. Since both T and S_{π_T} are π_T -Sylow subgroups they are equal, hence $\mathfrak{J} \subset \mathfrak{s}$. q.e.d.

16. SUBGROUP THEOREM. *If H is a closed subgroup of G and \mathfrak{J} is a Sylow system for H then there is a Sylow system \mathfrak{s} of G for which $\mathfrak{J} = \mathfrak{s} \cap H$, i.e., each $T \in \mathfrak{J}$ is $S \cap H$ for some $S \in \mathfrak{s}$.*

PROOF. Let $\{C'_p\}$ be the p -complements in \mathfrak{J} . For each $p \in \pi_H$ choose a p -complement C_p (in G) which contains C'_p . If $p \notin \pi_H$ choose a p -complement C_p containing H . Then the Sylow system generated by $\{C_p\}$ will do. Suppose $T \in \mathfrak{J}$. Then $S_{\pi_T} \cap H$ contains T and is a π_T group. $[H:T] = [H:S_{\pi_T} \cap H][S_{\pi_T} \cap H:T]$ and T a π_T -Sylow subgroup imply $[H:S_{\pi_T} \cap H]$ and π_T are coprime. Thus $S_{\pi_T} \cap H$ is a π_T -Sylow subgroup and therefore equals T . q.e.d.

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