THEOREM 4.2. *A complete regular ring S is continuous if and only if $\overline{S}_l = \overline{S}_r$.*

By [3, Theorem 3] every continuous regular ring is (both left and right) self injective if it contains no nonzero strongly regular ideals. Hence we have

THEOREM 4.3. *Let S be a complete regular ring, and suppose that S does not contain any nonzero strongly regular ideals. Then $\overline{S}_l = \overline{S}_r$ (if and) only if S is self injective, that is, $S = \overline{S}_l = \overline{S}_r$.*

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INVERSE LIMITS OF SOLVABLE GROUPS

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In this paper we generalize to groups of Galois type some results of P. Hall on finite solvable groups [1; 2; 3]. We need, in a modified form, some results of van Dantzig: the definition of supernatural numbers (which are related to van Dantzig's universal numbers) and Theorem 5, which he proved for ordinary p-Sylow subgroups [6]. Lemmas 1 and 4 and the method of proof in Theorem 5 are due to Tate [5].

A topological group G is of *Galois type* if it is compact and totally disconnected. In any Galois type group the open normal subgroups form a neighborhood base at the identity. Every closed subgroup is the intersection of the open subgroups containing it [4]. Whenever M and N are open normal subgroups of G and $N \supset M$ we shall write ϕ_N^M for the natural homomorphism of G/M onto G/N (these quotient groups are finite) and ϕ_N for the natural homomorphism of G onto G/N. G is the inverse limit of the groups $\{G/N\}$, N ranging over the open normal subgroups of G. Conversely, the inverse limit of finite groups is of Galois type.

A *supernatural number* is a formal product of powers of primes. Infinitely many factors are permitted, with exponents 1, 2, $\cdots$, ∞. Supernatural numbers are multiplied by addition of exponents. If π is a set of primes and δ a supernatural number we say

$$\delta \text{ and } \pi \text{ are coprime} \quad \text{when} \quad p \in \pi \text{ implies } p \nmid \delta,$$

$$\pi \mid \delta \ (\pi \text{ divides } \delta) \quad \text{when} \quad p \in \pi \text{ implies } p \mid \delta,$$

$$\delta \mid \pi \quad \text{when} \quad p \mid \delta \text{ implies } p \in \pi.$$

Whenever G is a group of Galois type and S is a closed subgroup of G we define the index of S in G ($[G:S]$) as the supernatural number which is the least common multiple of the integers $[G:U]$ as U ranges over open subgroups of G containing S. Then $[G:S]$ is also the l.c.m. $\{[G:SN]\}$ as N ranges over the open normal subgroups of G. The order of G, $o(G)$, is $[G:1]$. G is a π group if $o(G) \mid \pi$. A closed subgroup S of G is a π-Sylow subgroup if π and $[G:S]$ are coprime and S is a π group.

1. **LEMMA.** *If S and T are closed subgroups of the Galois type group G and S contains T then $[G:T] = [G:S][S:T]$.*

PROOF. For N an open normal subgroup of G we know $[G:TN] = [G:SN][SN:TN]$ and $[SN:TN] = [S:T(S\cap N)]$. Taking the least common multiples as N varies we see that $[G:T]$ divides $[G:S][S:T]$, so we need only show that $[G:S][S:T]$ divides $[G:T]$. Suppose that for m and $k < \infty$, p^m divides $[G:S]$ and p^k divides $[S:T]$. It follows that there are open normal subgroups N_1 and N_2 of G for which $p^m \mid [G:SN_1]$ and $p^k \mid [S:T(S\cap N_2)]$. Let $N = N_1 \cap N_2$. Then $p^m \mid [G:SN]$ and $p^k \mid [S:T(S\cap N)]$. Therefore p^{m+k} divides $[G:TN]$ and thus also divides $[G:T]$. Therefore $[G:S][S:T]$ divides $[G:T]$. q.e.d.

2. **COROLLARY.** *A subgroup S is a π-Sylow subgroup for some π if and only if its order and index are coprime.*

3. **COROLLARY.** *One π-Sylow subgroup cannot strictly contain another.*

PROOF. Let S and T be π-Sylow subgroups for which $T \subset S$. Then $o(S) = o(T)[S:T]$ implies $[S:T]$ divides π, but $[G:T] = [G:S][S:T]$ implies $[S:T]$ and π are coprime, which is a contradiction unless $S = T$.

4. **LEMMA.** *If S is the intersection of a decreasing family $\{S_\alpha\}$ of closed subgroups of G then $[G:S] = l.c.m. \{[G:S_\alpha]\}$.*

PROOF. Lemma 1 implies $[G:S] = [G:S_\alpha][S_\alpha:S]$ for all α, hence

l.c.m. $\{[G:S_\alpha]\}$ divides $[G:S]$. Conversely, if $m < \infty$ and $p^m | [G:S]$ then there is an open normal subgroup N for which $p^m | [G:SN]$. Since $G - SN$ is compact and disjoint from $S = \cap S_\alpha$ there is a finite intersection $S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}$ contained in SN. However $S_{\alpha_1} \cap \cdots \cap S_{\alpha_k}$ equals S_{α_k} since the S_α are decreasing. Therefore $p^m | [G:S_{\alpha_k}]$, hence $p^m | $l.c.m.$\{[G:S_\alpha]\}$. Thus $[G:S] = $l.c.m.$\{[G:S_\alpha]\}$. q.e.d.

A group of Galois type is *projectively solvable* if it is the inverse limit of finite solvable groups.

5. THEOREM. *Let G be projectively solvable. For each π dividing $o(G)$, G has π-Sylow subgroups, any two are conjugate, and any π subgroup of G is contained in one of them.*

PROOF. Let $\mathfrak{M}$ be the set of closed subgroups S of G for which $[G:S]$ and π are coprime. $G \in \mathfrak{M}$ and $\mathfrak{M}$ is partially ordered by inclusion. Lemma 4 shows that the intersection S of a maximal chain from $\mathfrak{M}$ is minimal in $\mathfrak{M}$. We shall show such an S is necessarily a π group.

Let N be an open normal subgroup of G. We must show $o(S/S\cap N)$ divides π. Since G/N is solvable so is $S/S\cap N$, which is isomorphic to SN/N. Choose a π-Sylow subgroup W of $S/S\cap N$ (Hall, [1]). Let T be the inverse image in S of W under $\phi_{S\cap N}$. T is closed in G and both open and closed in S. $[G:T] = [G:S][S:T]$. $[G:S]$ and π are coprime since $S \in \mathfrak{M}$. Since π and $[S/S\cap N:W]$, which equals $[S:T]$, are coprime, we see that $[G:T]$ and π are coprime. Therefore $T \in \mathfrak{M}$. Since $T \subset S$ and S is minimal in $\mathfrak{M}$, $T = S$. Therefore $S/S\cap N = T/S\cap N = W$, but we know $o(W)$ divides π. Therefore S is a π-Sylow subgroup of G.

Now let T be any π subgroup of G. For each open normal subgroup N of G the set K_N of members of G whose images σ under ϕ_N satisfy $\sigma^{-1}\phi_N(T)\sigma \subset \phi_N(S)$ is closed and nonempty (Hall, [1]). $N \subset M$ implies $K_N \subset K_M$, hence $\{K_N\}$ has the finite intersection property. If $\tau \in \cap K_N$ then $\tau^{-1}T\tau \subset S$. Therefore any π group has a conjugate inside a given π-Sylow subgroup. Corollary 3 then implies any two π-Sylow subgroups are conjugate. q.e.d.

6. COROLLARY. *Let $\mathfrak{N}(S)$ be the normalizer of the π-Sylow subgroup S. Then $[G:\mathfrak{N}(S)]$ and π are coprime since $S \subset \mathfrak{N}(S)$.*

Theorem 5 guarantees the existence of π-Sylow subgroups. Zorn's lemma was required in the proof. Once this existence has been established, however, the π-Sylow subgroups can be characterized somewhat more constructively.

7. THEOREM. *Each π-Sylow subgroup S of G may be written as the*

inverse limit of groups $\{S_N\}$ where S_N is a π-Sylow subgroup of G/N and ϕ_N^M maps S_M onto S_N. Conversely, if $\{S_N\}$ is a family of subgroups of the G/N with these properties then the inverse limit S of $\{S_N\}$ is a π-Sylow subgroup of G.

PROOF. Any closed subgroup S of G may be written as the inverse limit of $\{S_N\}$ where $S_N = SN/N$, which is isomorphic to $S/S \cap N$. Then ϕ_N^M maps S_M onto S_N. The argument in Theorem 5 shows that S_N is a π-Sylow subgroup of G/N if S is a π-Sylow subgroup of G.

Conversely, if each S_N is a π-Sylow subgroup then S is a π group, hence S is contained in some π-Sylow subgroup S'. Then $S'N/N$ must equal SN/N for all N, hence $S = S'$. q.e.d.

We can now generalize Hall's results on Sylow systems [3]. Let G be projectively solvable, π_G the set of primes dividing $o(G)$. Choose a $\pi_G - \{p\}$-Sylow subgroup (p-complement) C_p for each $p \in \pi_G$. For $\Sigma \subset \pi_G$ let $S_\Sigma = \bigcap_{p \notin \Sigma} C_p$. We set $S_{\pi_G} = G$. Then $\mathcal{S} = \{S_\Sigma\}$ is a *Sylow system* of G.

8. THEOREM. S_Σ is a Σ-Sylow subgroup of G.

PROOF. Each C_p is the inverse limit of p-complements of the groups G/N (Theorem 7). Hall's theorem [3] tells us that the intersections of the p-complements for $p \notin \Sigma$ are Σ-Sylow subgroups of the groups G/N. These Σ-Sylow subgroups are mapped onto each other by the ϕ_N^M, hence their inverse limit is a Σ-Sylow subgroup which is easily proved equal to S_Σ. q.e.d.

9. LEMMA. If $\{\Sigma_\alpha\}$ is a family of subsets of π_G then $S_{\cap \Sigma_\alpha} = \bigcap S_{\Sigma_\alpha}$.

This follows trivially from the definition of S_Σ.

10. THEOREM. $S_\Sigma S_\Delta = S_{\Sigma \cup \Delta}$.

PROOF. If $\sigma \delta \in S_\Sigma S_\Delta$ each of its projections $\phi_N(\sigma\delta)$ lies in the product of a Σ-Sylow subgroup and a Δ-Sylow subgroup of G/N (Theorem 7). By Hall's theorem [3] this product is a $\Sigma \cup \Delta$-Sylow subgroup of G/N, and is in fact $\phi_N(S_{\Sigma \cup \Delta})$. Hence $S_\Sigma S_\Delta \subset S_{\Sigma \cup \Delta}$.

To prove the reverse inclusion, suppose $\tau \in S_{\Sigma \cup \Delta}$. Then for each open normal subgroup N, $\phi_N(\tau) \in \phi_N(S_{\Sigma \cup \Delta})$, which equals $\phi_N(S_\Sigma)\phi_N(S_\Delta)$. Thus there are elements $\sigma_N \in S_\Sigma$ and $\delta_N \in S_\Delta$ for which $\phi_N(\sigma_N \delta_N) = \phi_N(\tau)$. This shows that the set K_N of those members η of $S_\Sigma S_\Delta$ for which $\phi_N(\eta) = \phi_N(\tau)$ is nonempty. K_N is closed since $S_\Sigma S_\Delta$ is closed, even though we do not yet know it is a subgroup. $\{K_N\}$ has the finite intersection property since for open normal subgroups $N_1, \cdots, N_j, \bigcap K_{N_i} \supset K_{\cap N_i}$. Choose $\rho \in \bigcap K_N$. For all N, $\phi_N(\rho) = \phi_N(\tau)$ implies $\rho = \tau$, but we know $\rho \in S_\Sigma S_\Delta$. Therefore $S_{\Sigma \cup \Delta} \subset S_\Sigma S_\Delta$. q.e.d.

11. COROLLARY. $S_\Sigma S_\Delta = S_{\Sigma \cup \Delta} = S_\Delta S_\Sigma$, *i.e.*, S *consists of commuting subgroups. In particular,* $\{S_{\{p\}}\}$ *is a commuting family of ordinary p-Sylow subgroups.*

12. COROLLARY. *Theorem* 10 *is true for infinite products, i.e.,* $[S_{\Sigma_\alpha} S_{\Sigma_\beta} \cdots]^- = S_{\cup \Sigma_\alpha}$ *for any family* $\{\Sigma_\alpha\}$ *of subsets of* π_G.

PROOF. By $S_{\Sigma_\alpha} S_{\Sigma_\beta} \cdots$ we mean the set of products of finitely many members of $\cup S_{\Sigma_\alpha}$. Since the S_{Σ_α} commute pairwise, this set, and hence its closure, S, is a subgroup. S is the smallest closed subgroup containing $S_{\Sigma_\alpha}^{\,\prime}$ for all α. Since $S_{\cup \Sigma_\alpha} \supset S_{\Sigma_\alpha}$ for all α we have $S_{\cup \Sigma_\alpha} \supset S$. It will suffice to prove that S is a $\cup \Sigma_\alpha$-Sylow subgroup, for then we may apply Theorem 8 and Corollary 3. S is a $\cup \Sigma_\alpha$ group since it is contained in one. Since $[G:S_{\Sigma_\alpha}]$ and Σ_α are coprime and $[G:S_{\Sigma_\alpha}]$ $= [G:S][S:S_{\Sigma_\alpha}]$ we see that for each α, $[G:S]$ and Σ_α are coprime. Thus $[G:S]$ and $\cup \Sigma_\alpha$ are coprime and S is a $\cup \Sigma_\alpha$-Sylow subgroup. q.e.d.

By combining the results 8 through 12 we prove

13. THEOREM. *For each projectively solvable group G there is a lattice injection of the set of subsets of* π_G *into the lattice of closed subgroups of G. Its range is commutative and consists of one Sylow subgroup of each type. The lattice join corresponds to set multiplication.*

We now study the family of Sylow systems. The Sylow system $\tau^{-1} S \tau$ generated by the τ-conjugates of the p-complements in S is said to be *conjugate* to S. If $S^* = \tau^{-1} S \tau$ then $S_\Sigma^* = \tau^{-1} S_\Sigma \tau$ for all $\Sigma \subset \pi_G$.

14. CONJUGACY THEOREM. *Any two Sylow systems* S, S^* *are conjugate.*

PROOF. For any finite set $\{p_1, \cdots, p_n\} \subset \pi_G$ there is a $\tau \in G$ for which $\tau^{-1} C_{p_i} \tau = C_{p_i}^*$, $i \leq n$. We prove this by induction. The proof is the same as Hall's [3]. For $n = 1$ the statement is Theorem 5. Assume that conjugation with τ takes C_{p_i} into $C_{p_i}^*$, $i \leq n-1$. Since $S_{p_n}^* C_{p_n}^*$ $= G = C_{p_n}^* S_{p_n}^*$ any conjugate of $C_{p_n}^*$ may be obtained by using an element of $S_{p_n}^*$. In particular, there is a $\sigma \in S_{p_n}^*$ for which $\sigma^{-1}(\tau^{-1} C_{p_n} \tau) \sigma$ $= C_{p_n}^*$. Since $S_{p_n}^* \subset C_p^*$ when $p \neq p_n$ we can conclude that $(\tau \sigma)^{-1} C_{p_i} (\tau \sigma)$ $= C_{p_i}^*$ for $i \leq n$.

Let K_p be the coset of the normalizer of C_p which takes C_p to C_p^*. K_p is closed. The induction above shows $\{K_p\}$ has the finite intersection property. Any $\tau \in \cap K_p$ clearly satisfies our requirements. q.e.d.

15. EMBEDDING THEOREM. *If* $\mathfrak{I}$ *is a commuting set of subgroups of G such that each* $T \in \mathfrak{I}$ *is a* π_T-*Sylow subgroup then there is a Sylow system* S *containing* $\mathfrak{I}$.

PROOF. For each $p \in \pi_G$ let T_p be the closure of the product of those $T \in \mathfrak{I}$ for which $p \nmid o(T)$. (If there are no such T let $T_p = \{e\}$.) The argument in Corollary 12 shows that T_p is a group. T_p is the inverse limit of the product in each G/N of the groups $\phi_N(T)$ for which $p \nmid o(T)$, therefore $p \nmid o(T_p)$. Using Theorem 5, choose a p-complement C_p containing T_p.

Let $\mathfrak{S}$ be the Sylow system generated by $\{C_p\}$. For each $T \in \mathfrak{I}$, S_{π_T} contains T since S_{π_T} is the intersection of the C_p for which $p \notin \pi_T$, i.e., for which $T \subset T_p \subset C_p$. Since both T and S_{π_T} are π_T-Sylow subgroups they are equal, hence $\mathfrak{I} \subset \mathfrak{S}$.   q.e.d.

16. SUBGROUP THEOREM. *If H is a closed subgroup of G and $\mathfrak{I}$ is a Sylow system for H then there is a Sylow system $\mathfrak{S}$ of G for which $\mathfrak{I} = \mathfrak{S} \cap H$, i.e., each $T \in \mathfrak{I}$ is $S \cap H$ for some $S \in \mathfrak{S}$.*

PROOF. Let $\{C_p'\}$ be the p-complements in $\mathfrak{I}$. For each $p \in \pi_H$ choose a p-complement C_p (in G) which contains C_p'. If $p \notin \pi_H$ choose a p-complement C_p containing H. Then the Sylow system generated by $\{C_p\}$ will do. Suppose $T \in \mathfrak{I}$. Then $S_{\pi_T} \cap H$ contains T and is a π_T group. $[H:T] = [H:S_{\pi_T} \cap H][S_{\pi_T} \cap H:T]$ and T a π_T-Sylow subgroup imply $[H:S_{\pi_T} \cap H]$ and π_T are coprime. Thus $S_{\pi_T} \cap H$ is a π_T-Sylow subgroup and therefore equals T.   q.e.d.

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