THE DIRECT PRODUCT OF RIGHT SINGULAR SEMIGROUPS AND CERTAIN GROUPOIDS[1]

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1. **Introduction.** Let G be a semigroup which satisfies the two axioms:

P 1.1. There is at least one (left identity) $e \in G$ such that $ea = a$ for all $a \in G$.

P 1.2. For every $a \in G$ and for every left identity $e \in G$, there is at least one $b \in G$ such that $ab = e$.

Such systems were investigated by A. H. Clifford [**1**] and H. B. Mann [**2**]. According to their results, G is the direct product of a group and a right singular semigroup, i.e., a semigroup in which $xy = y$ for all x, y. Clifford called such systems multiple groups, Mann called them (l, r) systems, but we call them right groups.

A generalization of right groups, namely, the direct product of a right singular semigroup and a semigroup with a two-sided identity, led us to a more general system in which a weakened associative law holds and to which we apply the name M-groupoid.

In this paper we prove that an M-groupoid is the direct product of a right singular semigroup and a groupoid with a two-sided identity and investigate how defining conditions for M-groupoids compare with those for right groups.

2. **Orthogonal decompositions of groupoids.** Let A and B be groupoids in Bruck's sense [**4**]. A groupoid S is the direct product of A and B, written $A \times B$, if and only if S is the set of all ordered pairs (a, b), $a \in A$, $b \in B$, under the binary composition $(a, b)(c, d) = (ac, bd)$.

A decomposition of a groupoid S is a partition of S, $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, $S_\alpha \cap S_\beta = \varnothing$, $\alpha \neq \beta$, in which for every $\alpha, \beta \in \Gamma$ there is $\gamma \in \Gamma$ such that $S_\alpha S_\beta \subset S_\gamma$. Two decompositions, $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, $S = \bigcup_{\lambda \in \Gamma'} T_\lambda$, of a groupoid S are said to be orthogonal if and only if for each $\alpha \in \Gamma$ and for each $\lambda \in \Gamma'$, $S_\alpha \cap T_\lambda = \{x_{\alpha\lambda}\}$, a set containing exactly one element.

Immediately we have the following lemma:

LEMMA 1. *A groupoid S is isomorphic to the direct product of two groupoids if and only if S has two orthogonal decompositions.*

Clifford introduced this notion and associated terminology in his paper but did not apply the principle directly. We use a restricted

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[1] The abstract of this paper was reported in [**5**].

version of this lemma which K. Shoda [3] established in the theory of direct decompositions using lattice theoretic methods. Our form of the principle is easily proved using elementary methods.

3. **The Main Theorem.** An M-groupoid S is a groupoid which satisfies the following conditions:

P 3.1. There is at least one $e\in S$ such that $ex=x$ for all $x\in S$.

P 3.2. If y or z is a left identity of S, then $(xy)z=x(yz)$ for all $x\in S$.

P 3.3. For any $x\in S$, there is a unique left identity e (which may depend on x) such that $xe=x$.

THEOREM 1. *An M-groupoid S is the direct product of a right singular semigroup and a groupoid with a two-sided identity, and conversely. The groupoid with the two-sided identity is obtained as Se where e is a left identity.*

PROOF. We find two orthogonal decompositions of S and apply Lemma 1. Let $e\in S$ be a left identity and consider Se. The mapping η: $S\to Se$, $s\eta=se$, is a surjective homomorphism. For any $se\in Se$,

$$(se)\eta = (se)e = s(ee) = se. \tag{3.4}$$

By P 3.1, P 3.2, $(xy)\eta=(xy)e=(x(ey))e=((xe)y)e=(xe)(ye)=(x\eta)(y\eta)$ for any $x, y\in S$. From the homomorphism η we obtain a decomposition of S:

$$S = \bigcup_{z\in Se} S_z, \quad S_{z_1}\cap S_{z_2} = \varnothing, \quad z_1\neq z_2; \quad S_z = \{x\in S: xe = z\}. \tag{3.5}$$

For $x\in S$ let $e_x\in S$ be the unique left identity such that $xe_x=x$, and consider the mapping β: $S\to R$, $x\beta=e_x$ where R is the subset of left identities. β is also a surjective homomorphism. If $xe_x=x$, $ye_y=y$, then, by P 3.2, $(xy)e_y=x(ye_y)=xy$. Thus, by P 3.3, $(xy)\beta=e_y=e_xe_y$ $=(x\beta)(y\beta)$. From the homomorphism we get the decomposition:

$$S = \bigcup_{u\in R} T_u, \quad T_{u_1}\cap T_{u_2} = \varnothing, \quad u_1\neq u_2; \quad T_u = \{x\in S: xu = x\}. \tag{3.6}$$

(3.5) and (3.6) are orthogonal. Let S_z and T_u be arbitrary cosets of the decompositions (3.5) and (3.6) respectively. Since $(zu)e=z(ue)$ $=ze=z$, where $z\in S_z$, and $(zu)u=z(uu)=zu$, $zu\in S_z\cap T_u$. By the definitions of S_z and T_u, $ye=z$ and $yu=y$ so that $y=yu=y(eu)$ $=(ye)u=zu$. Thus, $S_z\cap T_u$ contains exactly one element zu.

By (3.4), Se is a groupoid with a two-sided identity e. Also, R is a right singular semigroup. By Lemma 1, S is isomorphic to the direct product of R and Se.

Conversely, it follows easily that the direct product $R'\times S'$

$=\{(r, s): r\in R', s\in S'\}$, R' a right singular semigroup, S' a groupoid with two-sided identity, satisfies the definition.

4. Right groups. A right group S is a groupoid which satisfies:

P 4.1. For every $x, y, z\in S$, $(xy)z=x(yz)$.

P 4.2. For any $a, b\in S$, there is a unique $c\in S$ such that $ac=b$.

A right group is a special case of an M-groupoid.

THEOREM 2. *A right group S is isomorphic to the direct product of a right singular semigroup and a group.*

PROOF. It is sufficient to show that P 3.1 and P 3.3 are fulfilled and that Se, $e\in S$ is a left identity, is a group.

By P 4.2, for any $a\in S$, there is a unique $c\in S$ such that $ac=a$. Since S is a groupoid, we have, for any $x\in S$, that $(ac)x=ax$. By P 4.1, $a(cx)=ax$. An application of the uniqueness of P 4.2 to $a(cx)=ax$ yields $cx=x$ for all $x\in S$. Thus, $c\in S$ is a left identity and P 3.1, P 3.2, P 3.3 hold.

Next we shall prove that Se is a group. Clearly e is a right identity in Se. By P 4.2, for any $ae\in Se$, there is $c\in S$ such that $(ae)c=e$. But then, for $ce\in Se$, $(ae)(ce)=((ae)c)e=ee=e$. Thus ce is a right inverse of ae with respect to e. Now, Theorem 1 implies Theorem 2.

We now list some conditions which the elements of a groupoid S may satisfy, and from this list we formulate sets of conditions which when imposed on a groupoid S characterize a right group.

P 4.1. For all $a, b, c\in S$, $(ab)c=a(bc)$.

P 4.2. For all $a, b\in S$ there exists exactly one $c\in S$ such that $ac=b$.

P 4.3. $ab=ac$ implies $b=c$.

P 4.4. For all $a, b\in S$ there exists $c\in S$ such that $ac=b$.

P 4.5. There exists $e\in S$ such that for all $a\in S$, $ea=a$.

P 4.6. For all $a\in S$ and for all left identities $e\in S$ there is $c\in S$ such that $ac=e$.

P 4.7. For all $a\in S$ there is a left identity $e\in S$ and $c\in S$ such that $ac=e$.

P 4.8. For all $a\in S$ there is a left identity $e\in S$ and $c\in S$ such that $ca=e$.

THEOREM 3. *If S is a groupoid which satisfies any of the following sets of conditions, then S is a right group and conversely.*

I. $\{$P 4.1, P 4.2$\}$,

II. $\{$P 4.1, P 4.3, P 4.4$\}$,

III. $\{$P 4.1, P 4.4, P 4.5$\}$,

IV. $\{$P 4.1, P 4.5, P 4.6$\}$,

V. $\{$P 4.1, P 4.5, P 4.7$\}$,

VI. $\{P\ 4.1, P\ 4.5, P\ 4.8\}$,

VII. $S \cong R \times G$, *R is a right singular semigroup*, *G is a group.*

Although these formulations are proved equivalent in [1], we want to point out that this can be done in the following way.

I→VII→II→III→IV→V→VI→I.

We also have the following additional conditions:

P 4.9. If e and f are idempotents, then $ef=f$.

P 4.10. S is the set union of some groups; that is, $S=\bigcup_\alpha G_\alpha$, each G_α is a group.

VIII. $\{P\ 4.1, P\ 4.9, P\ 4.10\}$.

THEOREM 4. *A groupoid S satisfies* VIII *if and only if S is a right group.*

PROOF. Verification of VIII→VI and VII→VIII is the way we establish this.

VIII→VI. Let e_α be the identity of G_α and let $x \in S$ be an arbitrary element. By P 4.10, there is a β such that $x \in G_\beta$. Let e_β be the identity of G_β. By P 4.9,

$$e_\alpha x = e_\alpha(e_\beta x) = (e_\alpha e_\beta)x = e_\beta x = x. \tag{4.11}$$

Thus e_α is a left identity for S.

Again, by P 4.10, if $y \in S$, then there is γ such that $y \in G_\gamma$. Let $y^{-1} \in G_\gamma$ be the inverse of y with respect to the identity $e_\gamma \in G_\gamma$ then y^{-1} and e_γ satisfy P 4.8 since G_γ is a group.

VII→VIII. Since right singular semigroup R and group G are both associative, their direct product is associative. Thus, P 4.1 holds. The only idempotents in $R \times G$ are pairs of the form (y, e), where y is arbitrary and $e \in G$ is the identity. For such elements, $(x, e)(y, e) = (xy, ee) = (y, e)$ and P 4.9 holds. Any element of $R \times G$ has the form (y, a), $y \in R$, $a \in G$. For a fixed $y \in R$, let G_y be the subset of $R \times G$ in which the first element of each ordered pair is y. Clearly, G_y is isomorphic to G. Thus, $R \times G = \bigcup_{y \in R} G_y$ and P 4.10 holds.

COROLLARY 1. *If a semigroup S is the set union of some right groups S_α and $ef=f$ for all idempotents e and f, then S is a right group.*

This is an immediate consequence of Theorem 4.

5. **Conditions for M-groupoids.** In this section we investigate whether or not the characterizations of right groups in §4 when properly modified yield characterizations of M-groupoids. If it is not already included, adjoin "there is at least one left identity e" to the characterizations I through VI and replace associativity by the

weakened associative law P 3.2. Denote the new systems by I′ through VI′, respectively. The following counterexample shows that there are M-groupoids which do not satisfy I′ through VI′; that is, I′ through VI′ are not necessary conditions.

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	c	e	f	f
c	c	b	b	f	e	e
d	a	b	c	d	e	f
e	b	c	c	e	f	f
f	c	b	b	f	e	e.

With respect to sufficiency we have that I′ and II′ are sufficient conditions, IV′ through VI′ are not sufficient, but we have not yet answered this question for III′.

THEOREM 5. I′→{P 3.1, P 3.2, P 3.3}.

PROOF. It is only necessary to verify P 3.3. By P 4.2, for any $a\in S$ there is exactly one $f\in S$ such that $af=a$. For any left identity $e\in S$, $ae=(af)e=a(fe)$. By P 4.2, $ae=a(fe)$ yields $fe=e$. Now, for any $x\in S$, $fx=f(ex)=(fe)x=ex=x$. Thus, f is a left identity in S.

That II′→I′ is clear since P 4.3 and P 4.4 imply P 4.2.

The following example satisfies IV′, V′, VI′ but it is not an M-groupoid.

	a	b	c	d
a	a	b	c	d
b	b	a	a	a
c	b	a	a	a
d	b	a	a	a.

We now consider the following modifications of VIII.

P 5.1. Existence of left identity.

P 5.2. $(xy)z=x(yz)$ if y or z is a left identity.

P 5.3. If e, f are idempotents, $ef=f$.

P 5.4. S is the union of disjoint groupoids each of which has a two-sided identity.

P 5.5. There is a decomposition $\{S_\alpha\}$ of S such that each S_α is a groupoid with a two-sided identity.

VIII′. {P 5.1, P 5.2, P 5.3, P 5.4}.

VIII_2'. $\{$P 5.1, P 5.2, P 5.3, P 5.5$\}$.

VIII_1' does not imply $\{$P 3.1, P 3.2, P 3.3$\}$. The following example shows this.

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	c	b	c	c
c	c	b	b	c	b	b
d	a	b	c	d	e	f
e	e	f	f	e	f	f
f	f	e	e	f	e	e.

This multiplication table satisfies VIII_1', but it is not an M-groupoid since P 3.3 does not hold; that is, a, d are left identities for which $ba=b$ and $bd=b$.

However we have:

THEOREM 6. *VIII_2' characterizes an M-groupoid.*

PROOF. Necessity is clear since an M-groupoid is the direct product of a right singular semigroup and a groupoid with a two-sided identity according to Theorem 1.

For the proof of sufficiency, we may show that VIII_2' implies P 3.3. By the equalities of (4.11), a two-sided identity e_α of S_α is a left identity of S. Conversely, any left identity $e\in S$ is a left identity of some S_α and hence it coincides with the two-sided identity of S_α. Now, by P 5.5, for any $x\in S$, there is α such that $x\in S_\alpha$, and so $xe_\alpha=x$. Suppose that $xe_\beta=x$ for some left identity $e_\beta\in S$, where $e_\beta\in S_\beta$, $\beta\neq\alpha$. Immediately we have $S_\alpha S_\beta\subset S_\alpha$ by the assumption concerning decompositions; this contradicts P 5.3 since $e_\alpha e_\beta=e_\beta$. Thus P 3.3 holds.

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