ARITHMETIC MEANS OF FOURIER-STIELTJES-SINE-COEFFICIENTS

GÜNTHER GOES[1]

The theorem stated below contains improvements of statements of Hardy [4] and Kinukawa and Igari [6] and an interesting supplement to theorems of Fejér [1], [11, p. 107, 9.3] and Wiener [9], [11, p. 108, 9.6].

DEFINITIONS. Let L be the space of Fourier coefficients of Lebesgue integrable functions and dV the space of Fourier-Stieltjes-coefficients. Unambiguously let L and dV also denote the corresponding spaces of Fourier series and Fourier-Stieltjes-series respectively. Furthermore let

$$f \equiv \sum_{j=1}^{\infty} b_j \sin jt, \qquad \tilde{f} \equiv \sum_{j=1}^{\infty} b_j \cos jt.$$

In the following let E and E_1 be BK-spaces [2, p. 350] contained in dV. Then E_s and E_c are the spaces in E of sine- and cosine-coefficients respectively and $\tilde{E}=E_s \cap E_c$. If $b=\{b_j\}\in\tilde{E}$, $\|b\|_{E_s}=\|f\|_E$, $\|b\|_{E_c}=\|\tilde{f}\|_E$, then $\tilde{E}$ is a BK-space with the norm $\|b\|_{\tilde{E}}=\|b\|_{E_s}+\|b\|_{E_c}$ [10, p. 472]. Let

$$B_n = n^{-1}\sum_{j=1}^{n} b_j, \qquad B = \{B_n\},$$

and denote by T_H the mapping $b\to B$. $T_H\in(E, E_1)$ means $b\in E$ implies $B\in E_1$.

STATEMENTS. Hardy [4] proved $T_H\in(L_c, L_c)$ and $T_H\in(L_s, L_s)$ is also true [7], [3, Theorem 27]. Even the following can be proved.

THEOREM. *If $\sum_{j=1}^{\infty} b_j \sin jt$ is a Fourier-Stieltjes-series, then $\sum_{n=1}^{\infty} B_n \sin nt\in L$ and $\sum_{n=1}^{\infty} B_n \cos nt\in L$, where $B_n=n^{-1}\sum_{j=1}^{n} b_j$. Or in symbols: $T_H\in(dV_s, \tilde{L})$.*

SKETCH OF PROOF. T_H is a linear bounded transformation from L_s into L_s [10, p. 471] and $T_H\in(L_s, L_s)$ implies $\sup_n \|T_n\|<\infty$ [2, Theorem 4.4] where $T_n=\sup_{\|f\|_L\le 1}\|T_nf\|_L$ and $T_nf=\sum_{j=1}^{n}(1-j/(n+1))B_j \sin jt$.

Since dV is a norm determining manifold in L and since $\|f\|_L=\|f\|_{dV}$ for $f\in L$ we have also $\|T_n\|=\sup_{\|f\|_{dV}\le 1}\|T_nf\|_{dV}=O(1)$ $(n\to\infty)$

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and therefore $T_H \in (dV_s, dV_s)$ [2, Theorem 4.5]. (Correspondingly we get $T_H \in (dV_c, dV_c)$ but this is of no interest here.)

By Kinukawa and Igari [6, p. 274] we have $T_H \in (L_s, L_c)$ and since $T_H \in (L_s, L_s)$ we have $T_H \in (L_s, \tilde{L})$. The proof that $T_H \in (L_s, \tilde{L})$ implies $T_H \in (dV_s, d\tilde{V})$ is exactly the same as the proof that $T_H \in (L_s, L_s)$ implies $T_H \in (dV_s, dV_s)$. Since $d\tilde{V} = \tilde{L}$ [8; 11, p. 285] we have $T_H \in (dV_s, \tilde{L})$.

REMARKS. 1. Let V be the space of Fourier coefficients of functions of bounded variation. From the fact that $b \in \tilde{V}$ implies $\sum_{j=1}^{\infty} |b_j| < \infty$ [5; 11, p. 286] it follows with our theorem that $b \in dV_s$ implies $\sum_{n=1}^{\infty} n^{-2} | \sum_{j=1}^{n} b_j | < \infty$.

2. As remarked by Loo [7, p. 270] we have $T_H \notin (L_c, L_s)$.

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