ARITHMETIC MEANS OF FOURIER-STIELTJES-SINE-COEFFICIENTS

GÜNTHER GOES


DEFINITIONS. Let $L$ be the space of Fourier coefficients of Lebesgue integrable functions and $dV$ the space of Fourier-Stieltjes-coefficients. Unambiguously let $L$ and $dV$ also denote the corresponding spaces of Fourier series and Fourier-Stieltjes-series respectively. Furthermore let

$$f = \sum_{j=1}^{\infty} b_j \sin jt, \quad \tilde{f} = \sum_{j=1}^{\infty} b_j \cos jt.$$

In the following let $E$ and $E_i$ be BK-spaces [2, p. 350] contained in $dV$. Then $E_*$ and $E_e$ are the spaces in $E$ of sine- and cosine-coefficients respectively and $\tilde{E} = E_* \cap E_e$. If $b = \{ b_j \} \in E$, $\| b \|_{E_*} = \| f \|_{E_*}$, $\| b \|_{E_e} = \| \tilde{f} \|_{E_e}$, then $\tilde{E}$ is a BK-space with the norm $\| b \|_{\tilde{E}} = \| b \|_{E_*} + \| b \|_{E_e}$ [10, p. 472]. Let

$$B_n = n^{-1} \sum_{j=1}^{n} b_j, \quad B = \{ B_n \},$$

and denote by $T_H$ the mapping $b \rightarrow B$. $T_H \in (E, E_i)$ means $b \in E$ implies $B \in E_i$.

STATEMENTS. Hardy [4] proved $T_H \in (L_1, L_\infty)$ and $T_H \in (L_\infty, L_1)$ is also true [7], [3, Theorem 27]. Even the following can be proved.

THEOREM. If $\sum_{n=1}^{\infty} b_j \sin jt$ is a Fourier-Stieltjes-series, then $\sum_{n=1}^{\infty} b_n \sin nt \in L$ and $\sum_{n=1}^{\infty} B_n \cos nt \in L$, where $B_n = n^{-1} \sum_{j=1}^{n} b_j$. Or in symbols: $T_H \in (dV_*, L)$.

SKETCH OF PROOF. $T_H$ is a linear bounded transformation from $L_*$ into $L_\infty$ [10, p. 471] and $T_H \in (L_\infty, L_1)$ implies $\sup_n \| T_n \| < \infty$ [2, Theorem 4.4] where $T_n = \sup_{n \leq L \leq 1} \| T_n f \|_L$ and $T_n f = \sum_{j=1}^{n} (1 - j/(n+1)) B_j \sin jt$.

Since $dV$ is a norm determining manifold in $L$ and since $\| f \|_L = \| f \|_{dV}$ for $f \in L$ we have also $\| T_n \| = \sup_{n \leq \alpha \leq 1} \| T_n f \|_{dV} = O(1) \ (n \rightarrow \infty)$

Received by the editors December 14, 1961.

1 The research resulting in this paper was supported by the National Science Foundation (G 14876) and the National Research Council of Canada.

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and therefore \( T_H \in (dV_1, dV_2) \) [2, Theorem 4.5]. (Correspondingly we get \( T_H \in (dV_1, dV_2) \) but this is of no interest here.)

By Kinukawa and Igari [6, p. 274] we have \( T_H \in (L_1, L_2) \) and since \( T_H \in (L_1, L_2) \) we have \( T_H \in (L_2, L_1) \). The proof that \( T_H \in (L_2, L_1) \) implies \( T_H \in (dV_1, dV_2) \) is exactly the same as the proof that \( T_H \in (L_2, L_1) \) implies \( T_H \in (dV_1, dV_2) \). Since \( dV = L \) [8; 11, p. 285] we have \( T_H \in (dV_1, L) \).

Remarks. 1. Let \( V \) be the space of Fourier coefficients of functions of bounded variation. From the fact that \( F \in V \) implies \( \sum_{n=1}^{\infty} |b_n| < \infty \) [5; 11, p. 286] it follows with our theorem that \( b \in dV \) implies \( \sum_{n=1}^{\infty} n^{-1/2} \sum_{j=1}^{n} b_j < \infty \).

2. As remarked by Loo [7, p. 270] we have \( T_H \in (L_2, L_2) \).

References

4. G. H. Hardy, Notes on some points in the integral calculus. LXVI. The arithmetic mean of a Fourier constant, Mess. of Math. 58 (1928), 50–52.
8. F. Riesz and M. Riesz, Über die Randwerte einer analytischen Funktion, Quatrième congrès des math. scandinaves, Stockholm (1916), 27–44.