SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A NOTE ON CONTINUED FRACTIONS

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It is well known that the convergents p_n/q_n of a continued fraction of a real number α are its best approximations, i.e. that for every rational $a/b \neq p_n/q_n$ with $1 \leqq b \leqq q_n$ and $n \geqq 1$ there is

$$(1) \qquad |q_n\alpha - p_n| < |b\alpha - a|.$$

The usually produced proofs of this fact use rather intricate arguments.[1] Here is a proof of (1) based on the two following elementary properties of the continued fraction

$$\text{(i)} \qquad 1/q_{n+1} < |q_{n-1}\alpha - p_{n-1}| < 1/q_n,$$

$$\text{(ii)} \qquad q_n|q_{n-1}\alpha - p_{n-1}| + q_{n-1}|q_n\alpha - p_n| = 1.$$

If $a/b = p_{n-1}/q_{n-1}$ inequality (1) holds by (i):

$$|q_n\alpha - p_n| < 1/q_{n+1} < |q_{n-1}\alpha - p_{n-1}|.$$

If $|aq_{n-1} - bp_{n-1}| \geqq 1$ then

$$|a/b - \alpha| + |\alpha - p_{n-1}/q_{n-1}| \geqq |a/b - p_{n-1}/q_{n-1}| \geqq 1/(bq_{n-1})$$

i.e.

$$b|q_{n-1}\alpha - p_{n-1}| + q_{n-1}|b\alpha - a| \geqq 1,$$

while the assumption $1 \leqq b \leqq q_n$ implies by (ii)

$$1 \geqq b|q_{n-1}\alpha - p_{n-1}| + q_{n-1}|q_n\alpha - p_n|$$

whence

$$(1') \qquad |q_n\alpha - p_n| \leqq |b\alpha - a|.$$

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[1] See e.g. O. Perron, *Die Lehre von den Kettenbrüchen*, 1929, pp. 52–53; G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 3rd ed., 1956, pp. 151–152; I. Niven, *Irrational numbers*, 1956, pp. 62–63; J. W. S. Cassels, *An introduction to Diophantine approximations*, 1957, pp. 2–4; A. Ya. Khintchine, *Continued fractions* (in Russian), 2nd ed., 1949, pp. 38–40.

Equality in (1′) is for irrational α impossible in view of $a/b \neq p_n/q_n$ assumed. For rational α excluding equality in (1′) requires, strangely enough, additional argument which may run as follows. Substitute into (1′) with equality presumed

$$\alpha = \frac{P}{Q} = \frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}}$$

with rational r_n, and $Q \geqq q_n$, to get

$$|bP - aQ| = |r_n - a_n|,$$

where a_n is the nth partial quotient in the continued fraction expansion of α. The last equality shows that r_n is an integer, thus, by Euclid's algorithm, $r_n = a_n$ whence $a/b = P/Q = p_n/q_n$ contrary to assumption.

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