SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A NOTE ON CONTINUED FRACTIONS

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It is well known that the convergents \( p_n/q_n \) of a continued fraction of a real number \( \alpha \) are its best approximations, i.e. that for every rational \( a/b \neq p_n/q_n \) with \( 1 \leq b \leq q_n \) and \( n \geq 1 \) there is

\[
| q_n \alpha - p_n | < | ba - a | .
\]

The usually produced proofs of this fact use rather intricate arguments. Here is a proof of (1) based on the two following elementary properties of the continued fraction

(i) \( 1/q_{n+1} < | q_{n-1} \alpha - p_{n-1} | < 1/q_n \),

(ii) \( q_n | q_{n-1} \alpha - p_{n-1} | + q_{n-1} | q_n \alpha - p_n | = 1. \)

If \( a/b = p_{n-1}/q_{n-1} \) inequality (1) holds by (i):

\[
| q_n \alpha - p_n | < 1/q_{n+1} < | q_{n-1} \alpha - p_{n-1} | .
\]

If \( | aq_{n-1} - bp_{n-1} | \geq 1 \) then

\[
| a/b - \alpha | + | \alpha - p_{n-1}/q_{n-1} | \geq | a/b - p_{n-1}/q_{n-1} | \geq 1/(bq_{n-1})
\]

i.e.

\[
b | q_{n-1} \alpha - p_{n-1} | + q_{n-1} | b \alpha - a | \geq 1,
\]

while the assumption \( 1 \leq b \leq q_n \) implies by (ii)

\[
1 \geq b | q_{n-1} \alpha - p_{n-1} | + q_{n-1} | q_n \alpha - p_n |
\]

whence

\[
(1') | q_n \alpha - p_n | \leq | b \alpha - a | .
\]

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Equality in (1') is for irrational \( \alpha \) impossible in view of \( a/b \neq p_n/q_n \) assumed. For rational \( \alpha \) excluding equality in (1') requires, strangely enough, additional argument which may run as follows. Substitute into (1') with equality presumed

\[
\alpha = \frac{P}{Q} = \frac{p_{n-1}r_n + p_{n-2}}{q_{n-1}r_n + q_{n-2}}
\]

with rational \( r_n \), and \( Q \geq q_n \), to get

\[
|bP - aQ| = |r_n - a_n|,
\]

where \( a_n \) is the \( n \)th partial quotient in the continued fraction expansion of \( \alpha \). The last equality shows that \( r_n \) is an integer, thus, by Euclid's algorithm, \( r_n = a_n \) whence \( a/b = P/Q = p_n/q_n \) contrary to assumption.

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