# A NEW PROOF OF A COMPARISON THEOREM FOR ELLIPTIC EQUATIONS

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The classical Sturmian comparison theorem deals with solutions $u$, $v$ of two ordinary differential equations.

$$\frac{d}{dx}\left(a\frac{du}{dx}\right)+fu=0,$$

$$\frac{d}{dx}\left(b\frac{dv}{dx}\right)+gv=0,$$

where $a$ and $b$ are positive and continuously differentiable on a closed interval $I$ and $f$ and $g$ are continuous on $I$. The theorem states that if $a-b\geqq 0$ and $f\leqq g$ on $I$ and $x_1$, $x_2$ are zeros of $u$ in $I$, then $v$ has a zero in $[x_1, x_2]$.

This theorem was generalized to self-adjoint second order elliptic partial differential equations by Hartman and Wintner [1]. The purpose of this note is to present a new and simple proof of this generalization.

Let $u$ and $v$ be solutions of the elliptic equations

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)+fu=0,\qquad a_{ij}=\bar{a}_{ji}, \tag{1}$$

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(b_{ij}\frac{\partial v}{\partial x_i}\right)+gv=0,\qquad b_{ij}=\bar{b}_{ji} \tag{2}$$

in a bounded domain $R\subset E^n$. We assume that the $a_{ij}$ and $b_{ij}$ are of class $C'$ and that $f$ and $g$ are real and continuous in $\bar{R}$. The ellipticity of (1) and (2) requires that the hermitean matrices $A=(a_{ij})$ and $B=(b_{ij})$ be positive definite in $\bar{R}$.

THEOREM. *If $G$ is a bounded domain, $\bar{G}\subset R$, and*
(i) *$u=0$ on $\partial G$.*
(ii) *$A-B$ is non-negative definite in $\bar{G}$.*[1]
(iii) *$g\geqq f$ in $\bar{G}$.*
*Then $v$ must have a zero in $\bar{G}$.*

PROOF. Since $f$ is continuous in $\bar{R}$ we can choose a constant $c$ so

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[1] This is equivalent to the condition "$B^{-1}-A^{-1}$ is non-negative definite" which is used in [1].



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that $f(x)+c>0$ in $\bar{R}$. Let $F(x)=f(x)+c$. Define the operator

$$L = -\frac{1}{F}\sum \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)+\frac{c}{F}\cdot$$

Equation (1) states that $u$ is an eigenfunction of the boundary problem

$$(1') \qquad \begin{aligned} Lu &= \lambda u, \\ u &= 0 \text{ on } \partial G \end{aligned}$$

corresponding to the eigenvalue $\lambda=1$. Since the nodal lines of $u$ divide $G$ into a finite number of domains in which $u\neq 0$ it is sufficient to prove that the theorem holds for each such domain. In other words we may assume that $u\neq 0$ in $G$ and that $\lambda_1=1$.

Suppose $v\neq 0$ in $\bar{G}$. Then we can choose a domain $H$ satisfying $\bar{G}\subset H\subset R$ and $v\neq 0$ in $H$. Define

$$M = -\frac{1}{F}\sum \frac{\partial}{\partial x_j}\left(b_{ij}\frac{\partial}{\partial x_i}\right)+1-\frac{g}{F}.$$

Let $n$ denote the exterior normal to $\partial H$. Then, by an appropriate choice of a function $\sigma$ on $\partial H$, equation (2) states that $v$ is an eigenfunction of the boundary value problem

$$(2') \qquad \begin{aligned} Mv &= \nu v, \\ v+\sigma\frac{\partial v}{\partial n} &= 0 \text{ on } \partial H \end{aligned}$$

corresponding to the eigenvalue $\nu=1$. Since $v\neq 0$ in $H$, $\nu_1=1$. Since $\bar{G}\subset H$, classical variational principles guarantee that the first eigenvalue of the boundary problem

$$(2'') \qquad \begin{aligned} Mw &= \mu w, \\ w &= 0 \text{ on } \partial G \end{aligned}$$

will satisfy $\mu_1>\nu_1=1$. In particular, the assumption $v\neq 0$ in $\bar{G}$ has lead to the conclusion $\mu_1>\lambda_1$. We shall show that this conclusion is untenable.

Assuming

$$\int_G F|u|^2dx = \int_G F|w|^2dx = 1$$

the minimal property of $\mu_1$ yields

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$$\mu_1 = \int_G F\bar{w}Mw\,dx \leqq \int_G F\bar{u}Mu\,dx.$$

Writing

$$M = -\frac{1}{F}\sum \frac{\partial}{\partial x_j}\left(b_{ij}\frac{\partial}{\partial x_i}\right) + \frac{c}{F} + \left(1 - \frac{g+c}{f+c}\right)$$

and noting that

$$1 - \frac{g+c}{f+c} \leqq 0 \text{ in } G,$$

$$\sum b_{ij}\xi_i\xi_j \leqq \sum a_{ij}\xi_i\bar{\xi}_j \qquad \text{for all } (\xi_1, \cdots, \xi_n)$$

we have

$$\int_G F\bar{u}Mu\,dx = \int_G \left[\sum b_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \bar{u}}{\partial x_j} + c\,|u|^2 + F\left(1 - \frac{g+c}{f+c}\right)|u|^2\right] dx$$

$$\leqq \int_G \left[\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \bar{u}}{\partial x_j} + c\,|u|^2\right] dx = \lambda_1.$$

Thus we conclude that $\mu_1 \leqq \lambda_1$ and that $v(x) = 0$ for some $x$ in $\bar{G}$.

## Bibliography

1. P. Hartman and A. Wintner, *On a comparison theorem for self-adjoint partial differential equations of elliptic type*, Proc. Amer. Math. Soc. **6** (1955), 862.

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