ON THE w^*-SEQUENTIAL CLOSURE OF A CONE

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1. If X is a real Banach space and X^{**} the second conjugate space of X, then for each subset A of X^{**} let $K_X(A)$ be the w^*-sequential closure of A in X^{**}; thus $F\in K_X(A)$ if and only if there is a sequence $\{F_n\}$ in A such that $F(f)=\lim_n F_n(f)$ for all $f\in X^*$. If J_X is the canonical mapping from X into X^{**}, then $K_X(J_XX)$ is closed in the norm topology of X^{**} [1]. In the present paper it will be shown that if P is a norm-closed cone in X, then $K_X(J_XP)$ is norm-closed in X^{**}, but $K_X(K_X(J_XP))$ need not be norm-closed in X^{**}.

2. If s is a bounded real function on $[0, 1]$, let $\|s\|=\sup_{0\leq t\leq 1}|s(t)|$. If $S=\{s_k\}$ is a bounded sequence of functions, i.e., if each s_k is bounded and $\sup_{k\in\omega}\|s_k\|$ is finite, let $\|S\|=\sup_{k\in\omega}\|s_k\|$. If Q is a set of functions, let $L(Q)$ be the set of all functions x such that x is the pointwise limit of a bounded sequence in Q. Let $\mathfrak{A}$ be the collection of all double sequences $A=\{a_{ki}\}$ of non-negative numbers such that for each k the following conditions are satisfied: (1) $a_{ki}=0$ for all $i<k$ and for all but a finite number of $i\geq k$; and (2) $\sum_i a_{ki}=1$. If $S=\{s_k\}$ is a sequence of functions and $A\in\mathfrak{A}$, let T_AS be the sequence $\{s_k'\}$ such that $s_k'=\sum_i a_{ki}s_i$ for each k. It is clear that (1) T_A is a linear operator; (2) if S is bounded, then so is T_AS and $\|T_AS\|\leq\|S\|$; (3) if S is pointwise convergent, then T_AS is pointwise convergent to the same limit.

LEMMA 1. *If R and S are bounded sequences in the space $\mathcal{C}[0, 1]$ of continuous real functions on the interval $[0, 1]$ and R and S converge pointwise to functions r and s respectively, then for each $\epsilon>0$ there exists $A\in\mathfrak{A}$ such that $\|T_AR-T_AS\|<\|r-s\|+\epsilon$.*

PROOF. The sequence $R-S$ is bounded and converges pointwise to $r-s$. Hence by [1; proof of Lemma 1] there exists for each $\epsilon>0$ an A having the required property.

LEMMA 2. *Let P be a cone in $\mathcal{C}[0, 1]$, let $\{z_n\}$ be a sequence in $L(P)$, and let z be a bounded function such that $\lim_{n\to\infty}\|z_n-z\|=0$. Then $z\in L(P)$.*

PROOF. It may be assumed that $\|z_n-z\|<2^{-n}$ for each n. For each

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n there is a bounded sequence $S_n = \{s_{nk}\}$ in P which converges pointwise to z_n; by Lemma 1 it may be assumed that $\|S_n\| < \|z\| + 2^{-n}$.

By induction on n it can now be shown that for each n and for each $i \leqq n$ there exists a bounded sequence $S_{in} = \{s_{ink}\}$ in P such that

$$\lim_{k \to \infty} s_{ink}(t) = z_i(t) \qquad \text{for each } t \in [0, 1], \tag{1}$$

$$\lim_{k \to \infty} s_{ikk}(t) = z_i(t) \qquad \text{for each } t \in [0, 1], \tag{2}$$

and

$$\|S_{in} - S_{jn}\| < 2^{1-i} \qquad \text{if } i < j \leqq n. \tag{3}$$

For $n = 1$, the sequence S_{11} can be taken to be S_1. For $n > 1$, if $S_{i,n-1}$ has been obtained for each $i \leqq n-1$, Lemma 1 may be applied $(n-1)$ times in succession to obtain $A_1, \cdots, A_{n-1} \in \mathfrak{A}$ such that

$$\|T_{A_i} T_{A_{i-1}} \cdots T_{A_1}(S_{i,n-1} - S_n)\| < 2^{1-i} \tag{2.1}$$

for each $i < n$. The induction step is completed by letting $S_{in} = T_{A_{n-1}} \cdots T_{A_1} S_{i,n-1}$ for each $i < n$ and $S_{nn} = T_{A_{n-1}} \cdots T_{A_1} S_n$.

Now let $S = \{s_k\}$ be the bounded sequence in P such that $s_k = s_{kkk}$ for each k. Let $t \in [0, 1]$ and $i \in \omega$. Then for every $k \geqq i$,

$$\begin{aligned} &| s_k(t) - z(t) | \\ &\leqq | s_{kkk}(t) - s_{ikk}(t) | + | s_{ikk}(t) - z_i(t) | + | z_i(t) - z(t) | \\ &< 3 \cdot 2^{-i} + | s_{ikk}(t) - z_i(t) | . \end{aligned} \tag{2.2}$$

Since by construction $\lim_{k \to \infty} s_{ikk}(t) = z_i(t)$, it follows that $\lim_{k \to \infty} s_k(t) = z(t)$ for every t, so that $z \in L(P)$.

THEOREM 1. *If P is a norm-closed cone in a real Banach space X, then $K_X(J_X P)$ is norm-closed in X^{**}.*

PROOF. Let $F \in X^{**}$ be the limit in norm of a sequence $\{F_n\} \subset K_X(J_X P)$. Thus each F_n is the w^*-limit of a bounded [3, p. 209] sequence $J_X S_n$, where $S_n = \{s_{nk}\} \subset P$.

CASE 1. First suppose that X is a closed subspace of $\mathcal{C}[0, 1]$. For each $t \in [0, 1]$ let $f_t \in X^*$ be defined by $f_t(s) = s(t)$ for all $s \in X$. Then for each n it follows that $F_n(f_t) = \lim_{k \to \infty} s_{nk}(t)$. If z_n is defined on $[0, 1]$ by $z_n(t) = F_n(f_t)$, it follows that $z_n \in L(P)$ and moreover, since $\|f_t\| \leqq 1$ for each t, that $\|z_n - z_m\| \leqq \|F_n - F_m\|$ for all n and m, so that the function $z(t) = \lim_{n \to \infty} z_n(t)$ exists and belongs to $L(P)$ by Lemma 2. Thus there is a bounded sequence $\{s_k\} \subset P$ which converges pointwise to z. Since X is a subspace of $\mathcal{C}[0, 1]$, for each $f \in X^*$ there is a finite regular signed Borel measure μ_f on $[0, 1]$ such that $f(x)$

$=\int_0^1 x d\mu_f$ for every $x\in X$ [3, p. 397]. Hence

$$F_n(f) = \lim_{k\to\infty} f(s_{nk}) = \int_0^1 z_n d\mu_f \tag{2.3}$$

for every n, and therefore

$$F(f) = \lim_n \int_0^1 z_n d\mu_f = \int_0^1 z d\mu_f = \lim_{k\to\infty} \int_0^1 s_k d\mu_f = \lim_{k\to\infty} f(s_k), \tag{2.4}$$

so that F is the w^*-limit of $\{J_X s_k\}$.

CASE 2. If X is an arbitrary real Banach space, let Y be the norm-closed subspace and Q the norm-closed cone in X generated by $\{s_{nk}: n, k\in\omega\}$. Since Y is separable, there is an equivalence mapping E from Y onto a closed subspace Z of $\mathfrak{C}[0,1]$. Since each $F_n\in K_X(J_X P)$, for each n an element $G_n\in Y^{**}$ is unambiguously defined by

$$G_n(f|Y) = F_n(f) \qquad \text{for all } f\in X^*. \tag{2.5}$$

The sequence $\{G_n\}$ is clearly Cauchy since each $g\in Y^*$ has an extension $f\in X^*$ such that $\|f\|=\|g\|$. Since $\{G_n\}\subset K_Y(Q)$, it follows that the sequence $\{E^{**}G_n\}$ is a Cauchy sequence in $K_Z(J_Z EQ)$ converging in norm to $E^{**}G$. By Case 1 there is a bounded sequence $\{z_k\}\subset EQ$ such that $\{J_Z z_k\}$ is w^*-convergent to $E^{**}G$ in Z^{**}. Finally, it is straightforward to verify that the sequence $\{J_X E^{-1} z_k\}\subset J_X P$ is w^*-convergent to F in X^{**}.

3. For each $t_o\in[0, 2]$ and $q\in\omega$ let $f_{t_o q}\in\mathfrak{C}[0, 2]$ be defined by

$$f_{t_o q}(t) = \max[0, 1 - 2^q|t - t_o|] \qquad \text{for } t\in[0, 2]. \tag{3.1}$$

If p, i, and j are positive integers, let $s_{pi}=2^{-p}i$ and $t_j=2-2^{-j}$. Now if $a>1$ let X^a be the norm-closed subspace of $\mathfrak{C}[0, 2]$ generated by the set $G^a=\{x^a_{pq}: p, q\in\omega\}$, where

$$x^a_{pq}(t) = \max\left[\max_{1\le i<2^p} f_{s_{pi},p+q}(t), \max_{p\le j<p+q} a f_{t_i,j+q}(t)\right]. \tag{3.2}$$

LEMMA 3. *If P^a is the norm-closed cone in X^a generated by $\{x^a_{pq}: p, q\in\omega\}$, then the function x_o defined by*

$$x_o(t) = \lim_{p\in\omega}\left[\lim_{q\in\omega} x^a_{pq}(t)\right] \tag{3.3}$$

is an element of $L(L(P^a))$ with $\|x_o\|=1$, but if $\{y^h\}$ is a bounded sequence in $L(P^a)$ converging pointwise to x_o, then $\limsup_{h\in\omega}\|y^h\|\ge a$.

PROOF. It is trivial that $x_o\in L(L(P))$ and $\|x_o\|=1$. Indeed, x_o is the characteristic function of the set $D=\{s_{pi}: p\in\omega,\ i<2^p\}$. If $\{y^h\}$ is an arbitrary bounded sequence in $L(P^a)$ converging pointwise to x_o, then since the set of all finite linear combinations of the x^a_{pq} with non-negative coefficients is dense in P^a, each y^h is the pointwise limit of a bounded sequence $\{y^{hk}\}_{k=1}^{\infty}$, where each y^{hk} has the form

$$y^{hk}(t) = \sum_{p\geq 1;\, q\geq 1} a^{hk}_{pq} x^{a}_{pq}(t), \tag{3.4}$$

where each a^{hk}_{pq} is non-negative and for each pair (h, k) only a finite number of the a^{hk}_{pq} are positive. Without changing the value of $\limsup_{h\in\omega} \|y^h\|$, it may be assumed that $y^{hk}(\frac{1}{2})=1$ for all (h, k).

Let $\epsilon>0$ be given. For each $H\in\omega$ let

$$S_H = \{t \in [0, 1]: t \notin D,\ y^h(t) < \epsilon \text{ for all } h \geq H\}. \tag{3.5}$$

Since $\bigcup_{H\in\omega} S_H$ is of the second category in $[0, 1]$, there exists an H such that S_H is dense in a closed interval I. Choose an arbitrary $s_{p_oi_o}\in D\cap(\operatorname{int} I)$ with i_o odd; then there exists $H_o\geq H$ such that $y^h(s_{p_oi_o})>1-\epsilon$ for all $h\geq H_o$. Since there exist points of S_H between $s_{p_oi_o}$ and every s_{pi} such that $p<p_o$, it is clear that for every $h\geq H_o$ there exists K_h such that for every $k\geq K_h$,

$$\sum_{p\geq p_o;\, q\geq 1} a^{hk}_{pq} = y^{hk}(s_{p_oi_o}) - \sum_{p<p_o;\, q\geq 1} a^{hk}_{pq} x^{a}_{pq}(s_{p_oi_o}) > (1-\epsilon) - 2\epsilon. \tag{3.6}$$

Now fix $h\geq H_o$. Since y^h is a Baire function of the first class, there exists a point $\sigma_h\in I$ such that $y^h|I$ is continuous at σ_h [2, p. 143]; hence there is a closed interval $J\subset I$ such that $y^h(t)<2\epsilon$ for all $t\in J$. Choose $s_{p_1i_1}\in D\cap(\operatorname{int} J)$ with $p_1>p_o$; then there exists $K_h'\geq K_h$ such that for every $k\geq K_h'$,

$$\sum_{p\geq p_1;\, q\geq 1} a^{hk}_{pq} \leq y^{hk}(s_{p_1i_1}) < 2\epsilon. \tag{3.7}$$

Next, by (3.2), for every $t\in[0, 1]$ and every k,

$$\begin{aligned} y^{hk}(t) &\geq \sum_{p_o\leq p<p_1;\, q<p_1} a^{hk}_{pq}(1 - 2^{p+q}|t - s_{p_oi_o}|) \\ &\geq \sum_{p_o\leq p<p_1;\, q<p_1} a^{hk}_{pq}(1 - 2^{2p_1}|t - s_{p_oi_o}|). \end{aligned} \tag{3.8}$$

There exists $t_o\in S_H$ such that $|t_o-s_{p_oi_o}|<2^{-(2p_1+1)}$, and there exists $K_h''\geq K_h'$ such that $y^{hk}(t_o)<\epsilon$ whenever $k\geq K_h''$. Hence, by (3.8) with $t=t_o$,

$$(3.9)\qquad \sum_{p_0\leq p<p_1;\, q<p_1} a_{pq}^{hk} < \frac{\epsilon}{1-2^{2p_1}|t_0 - s_{p_0 i_0}|} < 2\epsilon$$

for all $k \geq K_h''$. Therefore, by (3.6), (3.7), and (3.9),

$$(3.10)\qquad \sum_{p_0\leq p<p_1;\, q\geq p_1} a_{pq}^{hk} > 1 - 7\epsilon$$

for all $k \geq K_h''$, so that $\|y^h\| \geq \lim_{k\to\infty} y^{hk}(t_{p_1}) \geq a(1-7\epsilon)$. Since ϵ may be an arbitrarily small positive number, it follows that $\limsup_{h\in\omega} \|y^h\| \geq a$.

THEOREM 2. *There exist a norm-closed subspace X of $\mathcal{C}[0, 1]$ and a norm-closed cone $P\subset X$ such that $K_X(K_X(J_XP))$ is not norm-closed in X^{**}.*

PROOF. For each $r\in\omega$ and each real function x defined on $[0, 2]$ let E_rx be the function defined on $[0, 1]$ by

$$(3.11)\qquad (E_rx)(t) = \begin{cases} x(2^{r+1}[t-2^{-r}]) & \text{if } 2^{-r} \leqq t \leqq 2^{1-r}, \\ 0 & \text{for all other } t. \end{cases}$$

Recalling the notation of Lemma 3 with $a = 2^{2r}$, observe that $E_r|X^{2^{2r}}$ is an equivalence mapping onto a subspace of $\mathcal{C}[0, 1]$, since $x(0) = x(2) = 0$ for all $x\in X^{2^{2r}}$. Let X be the closed subspace and P the closed cone in $\mathcal{C}[0, 1]$ generated by $\bigcup_{r\in\omega} \{E_rP^{2^{2r}}\}$. For each r it is clear that $E_rx_0\in L(L(P))$ and hence that an $F_r\in X^{**}$ can be unambiguously defined by

$$(3.12)\qquad F_r(f) = \int_0^1 (E_rx_0)\,d\mu_f \qquad \text{for all } f\in X^*;$$

then $F_r\in K_X(K_X(J_XP))$ and $\|F_r\| = 1$. Now let $F = \sum_{r=1}^{\infty} 2^{-r}F_r$; thus F belongs to the closure of $K_X(K_X(J_XP))$ in the norm topology.

Suppose $F\in K_X(K_X(J_XP))$. Then there is a bounded sequence $\{G_h\}\subset K_X(J_XP)$ whose w^*-limit is F. For each pair (r, h) let z_{rh} be the function on $[0, 2]$ defined by

$$(3.13)\qquad z_{rh}(t) = G_h(f_{2^{-r}+2^{-1-r}t}) \qquad \text{for } t\in[0, 2],$$

where $f_u\in X^*$ is defined by $f_u(x) = x(u)$ for $x\in X$ and real u. Then $\{z_{rh}\}_{h=1}^{\infty}$ is a bounded sequence in $L(P^{2^{2r}})$ which converges pointwise on $[0, 2]$ to $2^{-r}x_0$. By Lemma 3, $\limsup_{h\in\omega} \|z_{rh}\| \geq 2^r$, and hence $\limsup_{h\in\omega} \|G_h\| \geq 2^r$. Since r is an arbitrary positive integer, the sequence $\{G_h\}$ is unbounded, which gives a contradiction. Thus $F\notin K_X(K_X(J_XP))$ and the theorem is proved.

REMARK. The author has not been able to determine whether $K_X(K_X(J_X X))$ can fail to be norm-closed in X^{**}.

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