A GENERALIZATION OF FEJÉR’S PRINCIPLE
CONCERNING THE ZEROS OF
EXTREMAL POLYNOMIALS1

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Dedicated to Professor Einar Hille.

In 1922 Fejér set forth [1] a principle which has shown itself highly useful, to the effect that a polynomial \( p_n(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) \) which minimizes any classical norm in the complex plane such as

\[
\begin{align*}
(1) & \quad \max |p_n(z)|, \quad z \text{ on } E, \\
(2) & \quad \sum_{k=1}^{m} |p_n(z_k)|^p, \quad E: \{z_1, z_2, \cdots, z_m\}, \quad p > 0, \\
(3) & \quad \int_E |p_n(z)|^p |dz|, \quad E \text{ a Jordan arc or curve}, \quad p > 0,
\end{align*}
\]

on a closed bounded point set \( E \) containing at least \( n+1 \) distinct points, must have all its zeros in the convex hull of \( E \). More generally, the norms (1), (2), (3) may be replaced by any monotone norm, namely any norm that decreases whenever the polynomial \( p_n(z) \) is replaced by an underpolynomial \( q_n(z) = (z - \beta_1)(z - \beta_2) \cdots (z - \beta_n) \neq p_n(z) \); the latter term requires

\[
|q_n(z)| < |p_n(z)| \quad \text{on } E \text{ where } p_n(z) \neq 0,
\]

\[
q_n(z) = p_n(z) \quad \text{on } E \text{ where } p_n(z) = 0.
\]

Fejér’s principle is readily proved; if the zero \( \alpha_1 \) of \( p_n(z) \) lies exterior to the convex hull of \( E \), if \( \alpha \) is the point of the convex hull nearest \( \alpha_1 \), and if we set \( \alpha' = (\alpha + \alpha_1)/2 \), then the polynomial \( q_n(z) = (z - \alpha')p_n(z)/(z - \alpha_1) \) is an underpolynomial of \( p_n(z) \) on \( E \), so \( p_n(z) \) cannot minimize any monotonic norm on \( E \).

The object of the present note is to give what is essentially a generalization of Fejér’s principle. It applies to the minimization of the difference or quotient of two monotone norms of a polynomial on two disjoint point sets:

It is especially appropriate that this paper should be dedicated to Professor Einar Hille, in view of his now classical work on the complex zeros of solutions of differential equations.

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Theorem 1. Let $F(\alpha)$ be a functional of the complex variable $\alpha$ and of the closed bounded sets $E_1$ and $E_2$, which decreases whenever simultaneously $|z_1 - \alpha|$ decreases for all $z_1$ on $E_1$ and $|z_2 - \alpha|$ increases for all $z_2$ on $E_2$; if $F(\alpha)$ is a minimum, then $\alpha$ cannot lie on a line separating $E_1$ and $E_2$.

Suppose $\alpha$ lies on a line $L$ separating $E_1$ and $E_2$; let $\alpha'$ lie on the perpendicular to $L$ at $\alpha$ in the sense from $\alpha$ toward the side of $L$ on which $E_1$ lies, so that the distance $\alpha\alpha'$ is less than the distance from $L$ to $E_1$. By the Pythagorean theorem applied to suitable triangles whose legs are respectively parallel and perpendicular to $L$, it follows that $|z_1 - \alpha'| < |z_1 - \alpha|$ for all $z_1$ in $E_1$ and $|z_2 - \alpha'| > |z_2 - \alpha|$ for all $z_2$ in $E_2$, whence by the properties of $F(\alpha)$ we have $F(\alpha') < F(\alpha)$, so $F(\alpha)$ is not a minimum of the functional.

If there exists a line $L$ separating $E_1$ and $E_2$, there exist an infinity of them, and each such line separates a largest set $F_1$ containing $E_1$ from a largest set $F_2$ containing $E_2$; Theorem 1 asserts that $\alpha$ lies in $F_1$ or $F_2$.

As an immediate illustration of Theorem 1, we formulate

Theorem 2. Let the point sets $E_1$ and $E_2$ be disjoint, let $E_1$ consist of more than $n$ points, and let $\|P(z)\|_1$ and $\|P(z)\|_2$ be monotone norms on $E_1$ and $E_2$ respectively of the polynomial $P(z) = \prod_1^n (z - \alpha_i)$. Then no zero $\alpha_i$ of $P(z)$ can lie on a line separating $E_1$ and $E_2$ if $P(z)$ minimizes the functional

$$F(\alpha) = \|P(z)\|_1 - \|P(z)\|_2. \quad (5)$$

This functional clearly satisfies the conditions of Theorem 1. However, it may be pointed out that neither Theorem 1 nor Theorem 2 guarantees the existence or uniqueness of a minimum of the functional. For instance, let us choose $E_1: \{z = 1, 2\}$, $E_2: \{z = -1, -2\}$, $n = 1$, $P(z) = z - \alpha$, $\|P(z)\|_1 = \max \{P(z), z \text{ on } E_1\}$, $\|P(z)\|_2 = \exp \{\max |P(z)|, z \text{ on } E_2\}$; here the functional $F(\alpha) = F(\alpha_j)$ defined by (5) has no minimum. With the same definitions of $E_1$, $E_2$, $n$, $P(z)$, and $\|P(z)\|_1$, let us set $\|P(z)\|_2 = \max \{P(z), z \text{ on } E_2\}$, $F(\alpha) = F(\alpha_j)$ defined by (5); here $\min F(\alpha)$ occurs for all $\alpha \geq \frac{1}{2}$, $F(\alpha) = -3$.

Under the conditions of Theorem 1, it follows by Theorem 1 that if $E_1$ and $E_2$ lie on a line $L_1$, and if a point $A$ of $L_1$ separates $E_1$ and $E_2$ on $L_1$, then $F(\alpha)$ can be a minimum only if $\alpha$ lies on $L_1$ but does not separate $E_1$ and $E_2$ on $L_1$.

Theorem 1 contains Fejér’s principle, for we may choose $\|P(z)\|_1$ as any monotone norm on $E_1$, $\|P(z)\|_2$ as zero for every $P(z)$ and $E_2$, and define $F(\alpha) = F(\alpha_j)$ by (5).
A lemma is convenient in establishing another result.

**Lemma 1.** Let the point \( \alpha \) lie on a circle or line that separates the closed bounded point sets \( E_1 \) and \( E_2 \); then there exists a point \( \alpha' \) near \( \alpha \) such that we have

\[
\left| \frac{z_1 - \alpha'}{z_2 - \alpha'} \right| < \epsilon \left| \frac{z_1 - \alpha}{z_2 - \alpha} \right|, \quad 0 < \epsilon < 1,
\]

uniformly for all \( z_1 \) on \( E_1 \) and all \( z_2 \) on \( E_2 \).

Inequality (6) states that a certain cross-ratio is in modulus less than some \( \epsilon \), \( 0 < \epsilon < 1 \). If the plane is transformed by a linear transformation of the complex variable that carries \( \alpha \) to infinity, the given circle (or line) on which \( \alpha \) lies is transformed into a line \( L \) separating the images \( E'_1 \) and \( E'_2 \) of \( E_1 \) and \( E_2 \). There exists a circle (near the line \( L \)) containing all of \( E'_1 \) but no point of \( E'_2 \) in its interior, so if \( \alpha'_1 \) denotes the center of this circle we have

\[
\left| \frac{z_1 - \alpha'_1}{z_2 - \alpha'_1} \right| < \epsilon < 1
\]

uniformly for all \( z'_1 \) in \( E'_1 \) and all \( z'_2 \) in \( E'_2 \), where \( \epsilon \) is suitably chosen. The inverse of the preceding linear transformation carries \( \alpha'_1 \) into a point \( \alpha' \) satisfying (6). It may be noticed that \( \alpha' (\neq \alpha) \) can be chosen as near \( \alpha \) as desired, in such a way that \( \alpha \) and \( \alpha' \) are mutually inverse in a circle separating \( E_1 \) and \( E_2 \), where \( \alpha' \) and \( E_1 \) are separated by that circle from \( \alpha \) and \( E_2 \). Also, if \( \alpha \) lies at one of the two distinct intersections of two circles or lines \( L_1 \) and \( L_2 \) each of which separates \( E_1 \) and \( E_2 \), and if \( E_1 \) and \( E_2 \) lie respectively in two of the four regions into which \( L_1 + L_2 \) separates the plane having no arc of \( L_1 \) or \( L_2 \) as part of their common boundary, then \( \alpha' \) may be chosen near \( \alpha \) on the circle of the coaxal family determined by \( L_1 \) and \( L_2 \) bisecting the angle between \( L_1 \) and \( L_2 \), in direction from \( \alpha \) toward \( E_1 \).

We are now in a position to apply Lemma 1; Theorem 3 follows at once:

**Theorem 3.** Let \( F(\alpha) \) be a function of \( \alpha \), and of the closed bounded point sets \( E_1 \) and \( E_2 \), which decreases whenever

\[
\left| \frac{z_1 - \alpha}{z_2 - \alpha} \right|
\]

decreases simultaneously for all \( z_1 \) in \( E_1 \) and for all \( z_2 \) in \( E_2 \); if \( F(\alpha) \) is a minimum, then \( \alpha \) cannot lie on a circle or line separating \( E_1 \) and \( E_2 \).
If there exists a circle or line separating $E_1$ and $E_2$, there exist an
infinity of them, each of which separates a largest point set $F_1$ con-
taining $E_1$ from a largest point set $F_2$ containing $E_2$; Theorem 3 shows
that $\alpha$ lies in $F_1$ or in $F_2$.

We state explicitly an application of Theorem 3:

**Theorem 4.** Let us set

$$P(z) = \prod_{1}^{n} (z - \alpha_j), \quad F(\alpha_i) = \frac{\|P(z)\|_1}{\|P(z)\|_2},$$

where the norms are Tchebycheff norms (with positive weight functions
$\mu_1(z)$ and $\mu_2(z)$) on the closed bounded disjoint point sets $E_1$ and $E_2$, whereby $E_1$ contains more than $n$ points; if $F(\alpha_i)$ is a minimum, $\alpha_j$ cannot
lie on a circle or line separating $E_1$ and $E_2$.

If a zero, say $\alpha_1$, of $P(z)$ lies on a line or circle separating $E_1$ and $E_2$, and if $F(\alpha_1)$ is a minimum, the following algebraic inequalities
result from Lemma 1:

$$F(\alpha_1) = \max \left[ \mu_1(z_1) \left| z_1 - \alpha_1 \right| \cdot \prod_{2}^{n} \left| z_1 - \alpha_k \right|, z_1 \text{ on } E_1 \right],$$

$$\max \left[ \mu_2(z_2) \left| z_2 - \alpha_1 \right| \cdot \prod_{2}^{n} \left| z_2 - \alpha_k \right|, z_2 \text{ on } E_2 \right]$$

$$\leq \frac{\max \left[ \mu_1(z_1) \prod_{1}^{n} \left| z_1 - \alpha_k \right| \cdot \max \left| \frac{z_1 - \alpha_1}{z_1 - \alpha_1} \right| \right]}{\max \left[ \mu_2(z_2) \prod_{1}^{n} \left| z_2 - \alpha_k \right| \cdot \min \left| \frac{z_2 - \alpha_1}{z_2 - \alpha_1} \right| \right]} \leq \varepsilon \cdot F(\alpha_1),$$

$0 < \varepsilon < 1$, a contradiction that establishes Theorem 4.

It is not essential to suppose in Theorem 4 that Tchebycheff norms
are used, provided the norms are homogeneous of the same degree, in
the sense that for arbitrary positive continuous functions $\lambda_1(z)$ and
$\lambda_2(z)$ on $E_1$ and $E_2$ respectively we have for some $\rho(>0)$

$$\|\lambda_1(z)P(z)\|_1 \leq \max \lambda_1 \|P(z)\|_1, \quad \min \lambda_2 \|P(z)\|_2 \leq \|\lambda_2(z)P(z)\|_2.$$ 

For instance suppose $E_1$ and $E_2$ are rectifiable Jordan arcs or curves, and the norms are (as below) weighted $\rho$th and $q$th power norms respectively, $\rho > 0, q > 0$; if a zero $\alpha_1$ of $P(\alpha)$ lies on a line or circle
separating $E_1$ and $E_2$, and if $F(\alpha_1)$ is a minimum, we have
a contradiction as before.

It may be noticed that Theorems 2 and 4 overlap to a considerable extent; on the one hand, Theorem 2 refers to separation of $E_1$ and $E_2$ only by a line rather than a line or circle; on the other hand, the norms of Theorem 2 are arbitrary monotone norms, and if the functional of Theorem 4 is replaced by

\[
\log F(\alpha_j) = \log \|P(z)\|_1 - \log \|P(z)\|_2,
\]

we have essentially the difference of two particular monotone norms.

Under the conditions of Theorem 4, if $E_1$ and $E_2$ lie on a circle or line $L$, and if there exist circles separating them, then for the minimum functional all points $\alpha_j$ also lie on $L$. Whenever there exist disjoint circular regions containing $E_1$ and $E_2$ respectively, these regions contain all such $\alpha_j$. If $E_2$ consists of a single point $z_2$, all zeros of the polynomial $P(z)$ minimizing $F(\alpha_j)$ lie in the convex hull of $E_2$ with respect to $z_2$; here if $m (>0)$ denotes $\min F(\alpha_j)$ we have $\|P(z)\|_2 \leq \|P(z)\|_1/m$, which determines $\max |P(z_2)|$ over all $P(z)$ with prescribed $\|P(z)\|_1$; in this case the remark concerning the location of the $\alpha_j$ is due to Szegö [2, §5] and Fekete [3, p. 344].

The preceding results have all been established by considering a local variation of $\alpha$ (or $\alpha_j$); we proceed to consider the general question of a global variation of $\alpha$, and related results concerning maxima and minima. We use the same notation for a circular region (closed interior or exterior of a circle, or half-plane) as for its boundary, and shall prove

**Lemma 2.** In the extended plane, let the circular regions $C_1$ and $C_2$ be disjoint. Let $\alpha'$ denote the null circle in $C_1$ belonging to the coaxal family determined by the circles $C_1$ and $C_2$, and let $\lambda$ denote the ratio of the radius of the image of $C_1$ to the radius of the image of $C_2$ when $\alpha'$ is transformed to infinity. (i) If $\lambda > 3$, for all $z_1$ in $C_1$ and for all $z_2$ and $\alpha$ in $C_2$ inequality (6) holds uniformly with suitable $\epsilon (<1)$. (ii) If $\lambda \geq 3$, for all $z_1$ in $C_1$ and for all $z_2$ and $\alpha$ in $C_2$ we have
(iii) If $\lambda < 3$, no point $\alpha'$ exterior to $C_2$ exists for which (7) is valid for all $z_1$ in $C_1$ and for all $z_2$ and $\alpha$ in $C_2$, but (6) with $\alpha'$ as previously defined is valid for all $z_1$ in $C_1$, for all $z_2$ in $C_2$, and for each fixed $\alpha$ in a suitable subregion $C_3$ of $C_2$.

Since both (6) and (7) refer to the magnitude of a certain cross-ratio, it is no loss of generality to choose $\alpha'$ at infinity, and the circular regions $C_1$ and $C_2$ as $|z_1| \geq R_1$ and $|z_2| \leq R_2 (< R_1)$; we suppose too $|\alpha| \leq R_2$. The restrictions already made imply $|z_2 - \alpha| \leq 2R_2$, $|z_1 - \alpha| \geq R_1 - R_2$, so we have

$$|\frac{z_2 - \alpha}{z_1 - \alpha}| \leq \frac{2R_2}{R_1 - R_2} = \varepsilon$$

with $\varepsilon < 1$, thanks to our assumption $\lambda = R_1/R_2 > 3$. Inequality (8) is equivalent to (6) with $\alpha' = \infty$, which establishes (i). Part (ii) follows from (8) with $\varepsilon = 1$. The first part of (iii) is a consequence of the fact that with $1 < \lambda < 3$ and with the choices $z_1 = R_1$, $\alpha = R_2$, $z_2 = -R_2$ we have

$$|\frac{z_2 - \alpha}{z_1 - \alpha}| = \frac{2R_2}{R_1 - R_2} = \frac{2}{\lambda - 1} > 1;$$

for no choice of $\alpha'$ exterior to $C_2$ can we have

$$|\frac{z_1 - \alpha'}{z_2 - \alpha'}| \leq 1$$

for all $z_1$ in $C_1$ and all $z_2$ in $C_2$, so (7) is impossible. For the second part of (iii), we notice that without the requirement $|\alpha| \leq R_2$ we have $|z_2 - \alpha| \leq R_2 + |\alpha|$, $|z_1 - \alpha| \geq R_1 - |\alpha|$, and (6) is valid with $\varepsilon < 1$ provided we have $|\alpha| < R_1$ with

$$\frac{R_2 + |\alpha|}{R_1 - |\alpha|} \leq \varepsilon, \quad |\alpha| \leq \frac{\varepsilon R_1 - R_2}{1 + \varepsilon},$$

which is true for suitably chosen $\varepsilon$ whenever $|\alpha| < (R_1 - R_2)/2$, an inequality which obviously implies $|\alpha| < R_1$. It will be noted that the inequality $|\alpha| < (R_1 - R_2)/2$ restricts $\alpha$ to the interior of a certain circular region $C_3$ interior to $C_2$; such a fixed $\alpha$ may be replaced by $\alpha' = \infty$ with (6) valid.

We remark incidentally that (6) and (7) are considered valid even
with \( \alpha' = s_1 = \infty \), as is reasonable in view of the invariance of cross-ratio under linear transformation.

The significance of Lemma 2 with the hypothesis of Theorem 4 is as follows. Let \( E_1 \) and \( E_2 \) lie respectively in the (disjoint) circular regions \( C_1 \) and \( C_2 \) of Lemma 2. If \( F(\alpha_j) \) is a minimum, it follows by Theorem 4 that \( \alpha_j \) cannot lie exterior to both \( C_1 \) and \( C_2 \). Lemma 2 implies that in case (i) the point \( \alpha_j \) cannot lie in \( C_2 \), by precisely the application of (6) made in the proof of Theorem 4; in case (ii) with \( \lambda = 3 \), we may replace \( \alpha_j \) in \( C_2 \) by \( \alpha'_j \) in \( C_1 \) without increasing \( F(\alpha_j) \); in case (iii) the point \( \alpha_j \) for minimum \( F(\alpha_j) \) cannot lie interior to a specified circular region \( C_3 \) which is a subregion of \( C_4 \) bounded by a circle of the coaxal family determined by the circles \( C_1 \) and \( C_2 \).

Both Theorem 2 and Theorem 4 refer to the separation of \( E_1 \) and \( E_2 \) by lines and circles, and are therefore strongly reminiscent of Bôcher's theorem \([4, \S 4.2]\) to the effect that a finite point which lies on a line or circle separating the zeros and poles of a rational function \( R(z) \) cannot be a zero of the derivative \( R'(z) \). However, Bôcher's theorem is not related to any analogue of Lemma 2(i) and its application to Theorem 4. Indeed, Bôcher's theorem asserts that if two finite disjoint circular regions \( C_1 \) and \( C_2 \) contain respectively all zeros and all poles of the rational function \( R(z) \) of degree \( n \), and if the poles are all simple, then \( C_1 \) and \( C_2 \) contain each \( n - 1 \) zeros of \( R'(z) \); no zero of \( R'(z) \) can be displaced from \( C_2 \).

Theorems 2 and 4 consider the difference and the quotient of the norms of a polynomial on \( E_1 \) and \( E_2 \); likewise the sum and the product of two monotone norms on \( E_1 \) and \( E_2 \) may be considered, but the sum and product are themselves monotone norms on \( E_1 + E_2 \), and it follows by Fejér's principle that all zeros of a minimizing polynomial lie in the convex hull of \( E_1 + E_2 \).

It is not to be supposed that the present note exhausts the significance of the methods used. The reader may consider for instance the following:

I. Suppose \( E_1 \) and \( E_2 \) are closed bounded disjoint point sets, and that \( E_1 \) contains at least \( n + 1 \) points. If \( p_n(z) = (z - \alpha_1) \cdots (z - \alpha_n) \) and the functional

\[
\max\left[ \left| \frac{p_n(z)}{p_n(z)} \right|, \ z \text{ on } E_1 \right] \\
\min\left[ \left| \frac{p_n(z)}{p_n(z)} \right|, \ z \text{ on } E_2 \right]
\]

is least, the conclusion of Theorem 4 and possible application of Lemma 2 remain valid.

II. Under the same conditions on \( E_1 \) and \( E_2 \), and if \( E_2 \) also contains at least \( n + 1 \) points, and if the functional (9) with \( p_n(z) \) replaced by
$R(z) = \prod \frac{(z-\alpha_k)}{(z-\beta_k)}$ is least, no $\alpha_k$ or $\beta_k$ can lie on a circle or line separating $E_1$ and $E_2$; Lemma 2 also applies under suitable conditions.

III. Theorem 2 extends likewise to rational functions.

REFERENCES


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