THE RADIUS OF CONVEXITY FOR STARLIKE FUNCTIONS OF ORDER 1/2

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1. **Introduction.** Suppose that $f(z) = z + a_2 z^2 + \cdots$ is analytic for $|z| < 1$. The condition

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0 \quad \text{for } |z| < r \tag{1}$$

is necessary and sufficient for $f(z)$ to be univalent and convex for $|z| < r$ [**3**, p. 105, problem 108]. The condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{for } |z| < r \tag{2}$$

is necessary and sufficient for $f(z)$ to be univalent and starlike for $|z| < r$ [**3**, p. 105, problem 109]. Since each convex function is starlike a function which is convex in $|z| < 1$ satisfies (2) with $r = 1$. In fact, for functions convex in $|z| < 1$ (2) can be improved to

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad \text{for } |z| < 1 \tag{3}$$

[**2**; **6**]. In each of these references it is also shown that if $f(z)$ is convex in $|z| < 1$ then

$$\text{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \quad \text{for } |z| < 1. \tag{4}$$

As Strohhäcker has pointed out neither (3) nor (4) are sufficient for the convexity of $f(z)$ in $|z| < 1$. Indeed, a function may satisfy (4) without being univalent for $|z| < 1$.

In this paper we determine the largest circle $|z| < r$ such that each function satisfying (3) is convex in $|z| < r$. Also, we find the radius of univalence (and starlikeness) for functions satisfying (4).

Some properties of functions subject to (3) have been discussed in [**5**]. These functions are a particular case of the so-called starlike functions of order ρ which are defined by the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad \text{for } |z| < 1$$

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where $0<\rho<1$ [4].

2. **Lemma 1.** *The function* $g(z)$ *is analytic for* $|z|<1$ *and satisfies* $g(0)=1$ *and* $\operatorname{Re} g(z)>1/2$ *for* $|z|<1$ *if and only if* $g(z)=1/(1+z\phi(z))$ *where* $\phi(z)$ *is analytic and satisfies* $|\phi(z)|\leqq 1$ *for* $|z|<1$.

Proof. To prove one part of this lemma suppose that $g(z)=1+b_1z+\cdots$ is analytic and $\operatorname{Re} g(z)>1/2$ for $|z|<1$. Let $h(z)=2g(z)-1=1+2b_1z+\cdots$, $\operatorname{Re} h(z)>0$ for $|z|<1$. Let $k(z)=(1-h(z))/(1+h(z))$. Then, $k(z)$ is analytic and $|k(z)|<1$ for $|z|<1$, and $k(0)=0$. Therefore, $k(z)=z\phi(z)$, where $\phi(z)$ is analytic and satisfies $|\phi(z)|\leqq 1$ for $|z|<1$. Solving for $g(z)$ gives $g(z)=1/(1+z\phi(z))$.

The converse follows readily from the fact that $w=1/(1+z)$ maps $|z|<1$ onto $\operatorname{Re} w>1/2$.

The following lemma gives a way of constructing starlike functions of order 1/2 in terms of bounded functions.

Lemma 2. *The function* $f(z)$ *is analytic for* $|z|<1$ *and satisfies* $f(0)=0$, $f'(0)=1$, *and* $\operatorname{Re}\{zf'(z)/f(z)\}>1/2$ *for* $|z|<1$ *if and only if*

$$f(z)=z\exp\left\{-\int_0^z \frac{\phi(\sigma)}{1+\sigma\phi(\sigma)}\,d\sigma\right\}$$

where $\phi(z)$ *is analytic and* $|\phi(z)|\leqq 1$ *for* $|z|<1$.

Proof. To prove one-half of this lemma suppose that $f(z)=z+a_2z^2+\cdots$ is analytic and $\operatorname{Re}\{zf'(z)/f(z)\}>1/2$ for $|z|<1$. The function $f(z)/z$ is analytic and does not vanish for $|z|<1$. Applying Lemma 1 to the function $zf'(z)/f(z)$ gives $zf'(z)/f(z)=1/(1+z\phi(z))$ where $\phi(z)$ is analytic and satisfies $|\phi(z)|\leqq 1$ for $|z|<1$.

$$\frac{d(f(z)/z)/dz}{f(z)/z}=\frac{f'(z)}{f(z)}-\frac{1}{z}=-\frac{\phi(z)}{1+z\phi(z)}$$

$$\log\frac{f(z)}{z}=-\int_0^z\frac{\phi(\sigma)}{1+\sigma\phi(\sigma)}\,d\sigma$$

$$f(z)=z\exp\left\{-\int_0^z\frac{\phi(\sigma)}{1+\sigma\phi(\sigma)}\,d\sigma\right\}.$$

Conversely, suppose that $f(z)$ has such a representation. The analyticity of $f(z)$ for $|z|<1$ and $f(0)=0$ follow directly. Also, from this representation we obtain

$$f'(z) = \frac{1}{1+z\phi(z)} \exp\left\{-\int_0^z \frac{\phi(\sigma)}{1+\sigma\phi(\sigma)}\, d\sigma\right\},$$

$$\frac{zf'(z)}{f(z)} = \frac{1}{1+z\phi(z)}.$$

Then, according to Lemma 1, $\operatorname{Re}\{zf'(z)/f(z)\} > 1/2$ for $|z| < 1$. Also, $f'(0) = 1$.

THEOREM 1. *Suppose that* $f(z) = z + a_2 z^2 + \cdots$ *is analytic and satisfies* $\operatorname{Re}\{zf'(z)/f(z)\} > 1/2$ *for* $|z| < 1$. *Then,* $f(z)$ *maps* $|z| < (2(3^{1/2}) - 3)^{1/2}$ *onto a convex domain.*

PROOF. Applying Lemma 1 to the function $zf'(z)/f(z)$ gives

$$\frac{zf'(z)}{f(z)} = \frac{1}{1+z\phi(z)} \tag{5}$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leqq 1$ for $|z| < 1$. For such functions we have

$$|\phi'(z)| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \quad [1, \text{p. } 18]. \tag{6}$$

Differentiating (5) and multiplying through by $f(z)/f'(z)$ yields

$$\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)} = -\frac{f(z)}{f'(z)} \frac{\phi(z) + z\phi'(z)}{(1+z\phi(z))^2}.$$

Using (5) again gives

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{1 - z\phi(z) - z^2\phi'(z)}{1 + z\phi(z)}.$$

From this it follows that the condition (1) $\operatorname{Re}\{(zf''(z)/f'(z)) + 1\} > 0$ is equivalent to

$$\operatorname{Re}\{(1 - z\phi(z) - z^2\phi'(z))(1 + z\phi(z))^*\} > 0,^1$$

$$\operatorname{Re}\{1 - |z|^2|\phi(z)|^2 - z^2\phi'(z)(1 + z\phi(z))^*\} > 0,$$

$$\operatorname{Re}\{z^2\phi'(z)(1 + z\phi(z))^*\} < 1 - |z|^2|\phi(z)|^2. \tag{7}$$

From (6) we get

$$\begin{aligned}\operatorname{Re}\{z^2\phi'(z)(1 + z\phi(z))\}^* &\leqq |z^2\phi'(z)|\,(1 + |z|\,|\phi(z)|)\\ &\leqq \frac{|z|^2}{1 - |z|^2}(1 - |\phi(z)|^2)(1 + |z|\,|\phi(z)|).\end{aligned}$$

[1] Asterisks denote the conjugate of a complex number.

Therefore, (7) will be satisfied if

$$\frac{|z|^2}{1-|z|^2}(1-|\phi(z)|^2)(1+|z|\,|\phi(z)|)<1-|z|^2|\phi(z)|^2.$$

This inequality is equivalent to

$$(8)\qquad 2|z|^2+|z|(1-|z|^2)|\phi(z)|-|z|^2|\phi(z)|^2<1.$$

To discuss (8) let us consider the function $p(x)=2a^2+a(1-a^2)x-a^2x^2$ for $0\leqq x\leqq 1$, where $0<a<1(a=|z|,\ x=|\phi(z)|)$. If $0<a\leqq 2^{1/2}-1$ then $p(x)$ is increasing and has the maximum value $q(a)=a+a^2-a^3$. The function $q(a)$ also increases (for $0<a<1$) and $q(2^{1/2}-1)<1$. This implies that $p(x)<1$ for all x, $0\leqq x\leqq 1$, if $0<a\leqq 2^{1/2}-1$.

Now suppose that $a>2^{1/2}-1$. Then $p(x)$ achieves the maximum value of $(1/4)(1+6a^2+a^4)$ at $x=(1-a^2)/2a$. Since the inequality $(1/4)(1+6a^2+a^4)<1$ is equivalent to $a<(2(3^{1/2})-3)^{1/2}$ we can infer that $p(x)<1$ for all x, $0\leqq x\leqq 1$, if $2^{1/2}-1<a<(2(3^{1/2})-3)^{1/2}$.

We have shown: $p(x)<1$ for all x, $0\leqq x\leqq 1$, if $a<(2(3^{1/2})-3)^{1/2}$. Therefore, (8) is satisfied for every function $\phi(z)$ where $|\phi(z)|\leqq 1$ if $|z|<(2(3^{1/2})-3)^{1/2}$. Since (8) implies (1) this proves that $f(z)$ is convex in $|z|<(2(3^{1/2})-3)^{1/2}=0.68\cdots$.

Let us show that $f(z)$ need not be convex in the circle $|z|<r$ if $r>(2(3^{1/2})-3)^{1/2}$. Let $a=(2(3^{1/2})-3)^{1/2}$, $b=a/3^{1/2}$, $\phi(z)=(z-b)/(1-bz)$. Since $0<b<1$ this function maps $|z|<1$ onto $|\phi|<1$. A direct computation shows that $1-z\phi(z)-z^2\phi'(z)$ vanishes at $z=a$. A function $f(z)$ can be constructed which satisfies the hypotheses of the theorem and for which $zf'(z)/f(z)=1/(1+z\phi(z))$. Then $(zf''(z)/f'(z))+1$ vanishes at $z=a$ since

$$\frac{zf''(z)}{f'(z)}+1=\frac{1-z\phi(z)-z^2\phi'(z)}{1+z\phi(z)}\cdot$$

Therefore, $f(z)$ is convex in no circle $|z|<r$ with $r>a$.

Theorem 2. *If $f(z)=z+a_2z^2+\cdots$ is analytic and satisfies* $\mathrm{Re}\{f(z)/z\}>1/2$ *for* $|z|<1$ *then $f(z)$ is univalent and starlike for* $|z|<1/2^{1/2}$.

Proof. Applying Lemma 1 to $f(z)/z$ gives

$$(9)\qquad f(z)=\frac{z}{1+z\phi(z)}$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)|\leqq 1$ for $|z|<1$. Thus,

$|\phi'(z)| \leqq (1-|\phi(z)|^2)/(1-|z|^2)$. From (9) it follows that

$$\frac{zf'(z)}{f(z)} = \frac{1 - z^2\phi'(z)}{1 + z\phi(z)}\cdot$$

Let z be a complex number with $|z| < 1/2^{1/2}$. Then,

$$|z^2\phi'(z)| \leqq \frac{|z|^2}{1-|z|^2}(1 - |\phi(z)|^2) < 1 - |\phi(z)|^2 \leqq 1.$$

The number $1-z^2\phi'(z)$ lies in a circle with center at $w=1$ and with radius less than $1-|\phi(z)|^2$. Thus, $|\arg(1-z^2\phi'(z))| <$ Arc sin $(1-|\phi(z)|^2)$ for $|z| < 1/2^{1/2}$. Similarly, $|\arg(1+z\phi(z))| \leqq$ Arc sin $|z\phi(z)|$ for all z, $|z| < 1$.

Therefore, if $|z| < 1/2^{1/2}$

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\leqq |\arg(1 - z^2\phi'(z))| + |\arg(1 + z\phi(z))| \\ &< \operatorname{Arc\,sin}(1 - |\phi(z)|^2) + \operatorname{Arc\,sin}\frac{|\phi(z)|}{2^{1/2}} \\ &= \operatorname{Arc\,sin}\left\{\left(1 - \frac{|\phi(z)|^2}{2}\right)^{1/2}\right\} \\ &\leqq \operatorname{Arc\,sin} 1 \\ &= \frac{\pi}{2}\cdot\end{aligned}$$

This shows that $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < 1/2^{1/2}$. Therefore, $f(z)$ is univalent and starlike for $|z| < 1/2^{1/2}$.

To show that this result is a best-possible let us consider the function

$$f(z) = \frac{z}{1 + z\phi(z)} \quad \text{where} \quad \phi(z) = \frac{z - a}{1 - az} \quad \text{and} \quad a = \frac{1}{2^{1/2}}\cdot$$

Then, $|\phi(z)| < 1$ for $|z| < 1$, and $f(z)$ satisfies the hypotheses of this theorem. However, $f(z)$ is not univalent in $|z| < r$ if $r > a$ since a short computation shows that $f'(a) = 0$.

References

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LAFAYETTE COLLEGE

INVARIANT SUBSPACES IN L^1

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1. Denote by L^1 and L^2 the spaces of summable and square summable functions on the circle group and by H^1 and H^2 the subspaces of L^1 and L^2 consisting of those functions whose Fourier coefficients vanish for negative indices. A subspace M of L^1 or of L^2 is said to be invariant if

$$\chi \cdot M \subseteq M$$

and to be doubly invariant if also

$$\bar{\chi} \cdot M \subseteq M$$

where χ is the character

$$\chi(e^{ix}) = e^{ix}.$$

H^1 and H^2 are closed subspaces which are invariant but not doubly invariant.

Invariant subspaces on the circle were originally studied by Beurling [1] who showed that the closed invariant subspaces contained in H^2 are of the form $\phi \cdot H^2$ where ϕ is a function in H^2 which has modulus one a.e. Such functions are called inner functions. Rudin and de Leeuw [3, p. 476] have shown that the closed invariant subspaces in H^1 have the same structure as those in H^2. That is to say, they are of the form $\phi \cdot H^1$ where again ϕ is an inner function. The arguments given in [1; 3] depend to a considerable extent on the function theory of the spaces H^1 and H^2.

Recently, Helson and Lowdenslager [2] using Hilbert space methods have given a very elegant and simple proof of Beurling's theorem.

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