NOTE ON COHOMOLOGY RING OF CERTAIN SPACES

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1. **Statement of results.** The purpose of this note is to survey the structure of the cohomology ring of a simply connected topological space X having only nontrivial cohomology groups $H^{in}(X, Z_p) \approx Z_p$ for $i=0, 1, 2, \cdots, k$.

For such a space X, there exist always a cell complex $K=S^n \cup e^{2n} \cup \cdots \cup e^{kn}$ and a continuous mapping f of K into X such that f induces a ring isomorphism $f^*: H^*(X, Z_p) \approx H^*(K, Z_p)$. So we shall devote to consider the case $X=K$, and in particular $X=K=S^n \cup e^{2n} \cup e^{3n}$.

We consider that all cells of the complex $K=S^n \cup e^{2n} \cup e^{3n}$ have orientations (generators)

$$u \in H^n(K, Z), \qquad v \in H^{2n}(K, Z) \quad \text{and} \quad w \in H^{3n}(K, Z).$$

Then the ring structure of $H^*(K, Z)$ is completely determined by two integers a and b such that

$$u^2 = av \quad \text{and} \quad uv = bw.$$

We shall say that an oriented complex $K=S^n \cup e^{2n} \cup e^{3n}$ has a type (a, b) if the above relations hold.

Our main results are stated as follows.

THEOREM 1. *A complex $K=S^n \cup e^{2n} \cup e^{3n}$ of type (a, b) exists if and only if one of the following conditions are satisfied.*

(i) *n is odd and $a=0$.*

(ii) *$n=2$ or $n=4$.*

(iii) *n is even, $n\neq 2, 4, 8$, $a \equiv 0 \pmod 2$ and $ab \equiv 0 \pmod 3$.*

(iv)$_0$ *$n=8$, $a \equiv 0 \pmod 2$ and $ab \equiv 0 \pmod 3$.*

(iv)$_1$ *$n=8$, $a \equiv 1 \pmod 2$ and $ab \equiv 0 \pmod{24}$.*

THEOREM 2. *There exists no simply connected space X having the truncated polynomial ring $Z_p[u]/(u^{k+1})$ as its cohomology ring unless* $\dim u = 2, 4$, *where $p=2, 3$ and $k \geqq 3$.*

2. **Lemmas.** Let $\alpha \in \pi_{2n-1}(S^n)$ be the homotopy class of the attaching map of e^{2n}. Then it is well-known that the Hopf invariant $H(\alpha)$ of α is $\pm a$ (cf. [11]), and the following lemma is the same as Theorem 1.2 of [5].

LEMMA 1. *A complex $K=S^n \cup e^{2n} \cup e^{3n}$ of type (a, b) exists if and*

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only if there exists an element α *of* $\pi_{2n-1}(S^n)$ *such that* $H(\alpha)=a$ *and* $b[\alpha, \iota_n]\in\alpha\circ E^{n-1}\pi_{2n-1}(S^n)$.

If there is such an element α, then the element $k\alpha$ satisfies $H(k\alpha)=ka$ and $hb[k\alpha, \iota_n]\in k\alpha\circ E^{n-1}\pi_{2n-1}(S^n)$, for arbitrary integers k and h. Thus,

COROLLARY. *If there exists a complex of type* (a, b), *then there exist also complexes of types* (ka, hb) *for arbitrary integers* k *and* h.

From the general expansion formula, for Whitehead product $[\alpha\circ\gamma, \beta]$, given by Barcus-Barratt [2], we have particularly

LEMMA 2. $[\alpha, \iota_n]=[\iota_n, \iota_n]\circ E^{n-1}\alpha-H(\alpha)[[\iota_n, \iota_n], \iota_n]$ *for* $\alpha\in\pi_{2n-1}(S^n)$.

Here, H satisfies $H[\iota_n, \iota_n]=1+(-1)^n$, changing H in sign if it is necessary.

LEMMA 3. *There exists a complex of type* $(0, 1)$ *for arbitrary* n. *If* n *is even, then there exists complexes of types* $(2, 3)$ *and* $(6, 1)$. *If* $n=2$ *or* 4, *then there exists a complex of type* $(1, 1)$.

In fact, the product $S^n\times S^{2n}$ and complex (quaternion) projective 3-space are first and last examples. By Jacobi's identity for Whitehead product [8], $3[[\iota_n, \iota_n], \iota_n]=0$. If n is even, then $H([\iota_n, \iota_n])=2$. These relations prove the second assertion by setting $\alpha=[\iota_n, \iota_n]$ or $=3[\iota_n, \iota_n]$.

LEMMA 4. *Let* n *be even and* $n\geqq 4$. *There exists an element* $\theta\in\pi_{3n-3}(S^{n-1})$ *of order* 3 *such that* $E\theta=[[\iota_n, \iota_n], \iota_n]$.

PROOF. In this case the mod 3 Hopf invariant: $\pi_{3n}(S^{n+1})\to Z_3$ is trivial [6; 10]. Then $[[\iota_n, \iota_n], \iota_n]\neq 0$ by Proposition 2.5 of [13]. Since $3[[\iota_n, \iota_n], \iota_n]=0$, then $[[\iota_n, \iota_n], \iota_n]$ is of order 3. According to Serre [9], we have an isomorphism $\pi_{3n-2}(S^{2n-1})+\pi_{3n-3}(S^{n-1})\to\pi_{3n-2}(S^n)$ of the odd components. Then there are (uniquely) elements $\beta\in\pi_{3n-2}(S^{2n-1})$ and $\theta\in\pi_{3n-3}(S^{n-1})$ such that

$$[[\iota_n, \iota_n], \iota_n] = [\iota_n, \iota_n]\circ\beta + E\theta,\ 3\beta = 0 \text{ and } 3\theta = 0.$$

According to G. W. Whitehead [16], we compute $(\iota'+\iota'')\circ\gamma-\iota'\circ\gamma-\iota''\circ\gamma=\Gamma(\gamma)$, where ι' and ι'' denote the homotopy class of the canonical injections: $S^n\to S^n\vee S^n$ to the first and second factors. We have

$$\Gamma([\iota_n, \iota_n]\circ\beta) = \Gamma([\iota_n, \iota_n])\circ\beta = [\iota', \iota'']\circ 2\beta$$

since β is a suspension element. Also we have

$$\begin{aligned} \Gamma(E\theta) &= 0, \\ \Gamma([[\iota_n, \iota_n], \iota_n]) &= [[\iota', \iota'], \iota''] + [[\iota'', \iota'], \iota'] + [[\iota', \iota''], \iota'] \\ &\quad + [[\iota'', \iota''], \iota'] + [[\iota', \iota''], \iota''] + [[\iota'', \iota'], \iota''] \\ &= 0 \text{ (Jacobi's identity).} \end{aligned}$$

Thus $[\iota', \iota''] \circ 2\beta = 0$. The correspondence: $\alpha \rightarrow [\iota', \iota''] \circ \alpha$ is an injection of $\pi_{3n-2}(S^{2n-1})$ onto a direct factor of $\pi_{3n-2}(S^n \vee S^n)$ [4], thus we have $2\beta = 0$. Since $3\beta = 0$, we have $\beta = 0$ and $[[\iota_n, \iota_n], \iota_n] = E\theta$. Obviously θ is an element of order 3. q.e.d.

LEMMA 5. *Let n be even, $n \geqq 6$ and let $m = n/2$, then $E^{n-m-1}: \pi_{m+n-1}(S^m) \rightarrow \pi_{2n-2}(S^{n-1})$ is a homomorphism onto the odd component of $\pi_{2n-2}(S^{n-1})$.*

PROOF. Consider the exact sequence

$$\pi_{2k+n-2}(S^{2k-1}) \xrightarrow{E^2} \pi_{2k+n}(S^{2k+1}) \xrightarrow{j_*} \pi_{2k+n-2}(\Omega^2(S^{2k+1}), S^{2k-1}).$$

It is sufficient to prove that the above E^2 is a homomorphism onto of the p-primary components for $k \geqq [(m+1)/2] \geqq 2$ and for odd prime p. By [7; 12], the p-component of

$$\pi_{2k+n-2}(\Omega^2(S^{2k+1}), S^{2k-1}) = \begin{cases} 0 & \text{for } 2k + n - 2 < 2pk - 2, \\ Z_p & \text{for } 2k + n - 2 = 2pk - 2. \end{cases}$$

The condition $k \geqq [(m+1)/2]$ implies $2k+n-2 = 2k+2m-2 < 2k+2(p-1)k-2 = 2pk-2$ unless the case $p=3$ and $m=2k$. In the case $p=3$ and $m=2k$, the homomorphism j_* is equivalent to the mod 3 Hopf invariant: $\pi_{6k}(S^{2k+1}) \rightarrow Z_3$. Since $k \geqq 2$, the mod 3 Hopf invariant is trivial and thus j_* is a homomorphism which is trivial on 3-component. Therefore, in all cases, we have that E^2 is a homomorphism onto of the odd components, and thus the lemma has been proved. q.e.d.

COROLLARY. *Let n be even and $n \geqq 6$. For an arbitrary element β of $\pi_{2n-2}(S^{n-1})$, the composition $\beta \circ E^{n-1}\beta$ is contained in 2-component of $\pi_{3n-3}(S^{n-1})$.*

PROOF. By Lemma 5, $\beta = \beta_1 + \beta_2$ and $\beta_1 = E^{n-m-1}\beta'$ for an element β' of odd components and an element β_2 of 2-component. Obviously, $\beta_1 \circ E^{n-1}\beta_2 = \beta_2 \circ E^{n-1}\beta_1 = 0$ and $\beta_2 \circ E^{n-1}\beta_2$ is contained in 2-component. By use of the reduced join $\beta' \# \beta'$, we have that $2(E\beta_1 \circ E^n\beta_1) = 2(\beta' \# \beta') = 0$, i.e., the anti-commutativity of the composition [3]. By Serre's decomposition of $\pi_{3n-2}(S^n)$, the suspension homomorphism $E: \pi_{3n-3}(S^{n-1}) \rightarrow \pi_{3n-2}(S^n)$ has the kernel of 2-torsion. It follows that $\beta_1 \circ E^{n-1}\beta_1$ is contained in 2-component. Consequently, we have that

$\beta \circ E^{n-1}\beta = \beta_1 \circ E^{n-1}\beta_1 + \beta_2 \circ E^{n-1}\beta_2$ is contained in 2-component. q.e.d.

Recall the following results (cf. [15]).

$$\pi_{14}(S^7) \approx Z_{120},$$
$$\pi_{15}(S^8) = Z + E\pi_{14}(S^7) \approx Z + Z_{120},$$
$$\pi_{n+7}(S^n) = E^{n-8}\pi_{15}(S^8) \approx Z_{240} \qquad \text{for } n \geqq 9.$$

The kernel of $E: \pi_{15}(S^8) \to \pi_{16}(S^9)$ is generated by the Whitehead product $[\iota_8, \iota_8]$.

LEMMA 6. *There are elements $\sigma \in \pi_{15}(S^8)$ and $\tau \in \pi_{14}(S^7)$ such that $H(\sigma) = 1$, $[\iota_8, \iota_8] = 2\sigma - E\tau$ and the order of τ is 120. Thus σ and $E\tau$ generate $\pi_{15}(S^8)$, and for $k \geqq 1$, $E^{k+1}\tau = 2E^k\sigma$.*

In fact, there exists an element σ of $\pi_{15}(S^8)$ such that $H(\sigma) = 1$ and $E\sigma$ is of order 240. Then τ is determined by the formula $E^2\tau = 2E\sigma$.

Remark that we may handle the element σ as the class of Hopf fibre map. In particular, the correspondence $(\alpha, \beta) \to \sigma \circ \alpha + E\beta$ gives an isomorphism:

$$\pi_i(S^{15}) + \pi_{i-1}(S^7) \approx \pi_i(S^8).$$

Finally, we introduce the following result from Theorem 10.3 of [15].

LEMMA 7. *The order of the composition $\tau \circ E^6\sigma \in \pi_{21}(S^7)$ is* 8.

Since $2\tau \circ E^6\sigma = \tau \circ E^7\tau$ belongs to 2-component, by the corollary of Lemma 5, we may replace in Lemma 7 τ with σ' of [15].

3. Proof of theorems.

PROOF OF THEOREM 1.

(i) The case n is odd. By the anti-commutativity of the cup product, we have $2u^2 = 2av = 0$. Thus a has to be zero. For arbitrary integer b, a complex K of the type $(0, b)$ exists since Lemma 3 and the corollary of Lemma 1.

(ii) The case $n = 2$ or $n = 4$. The theorem is obvious by Lemma 3 and the corollary of Lemma 1.

(iii) The case n is even and $n \neq 2, 4, 8$. If a pair (a, b) of integers satisfies $a \equiv 0 \pmod 2$ and $ab \equiv 0 \pmod 3$, then there exists a complex K of type (a, b) since Lemma 3 and the corollary of Lemma 1.

Conversely assume that there exists a complex K of type (a, b) and apply Lemma 1. Then there exists an element $\alpha \in \pi_{2n-1}(S^n)$ such that $H(\alpha) = a$ and $b[\alpha, \iota_n] \in \alpha \circ E^{n-1}\pi_{2n-1}(S^n)$. By the nonexistence of the element of Hopf invariant 1 in $\pi_{2n-1}(S^n)$, $n \neq 2, 4, 8$ [1], we have that $a \equiv 0 \pmod 2$. Let $a = 2d$. $H(\alpha) = 2d = H(d[\iota_n, \iota_n])$. Then it follows from the exactness of the sequence

$$\pi_{2n-2}(S^{n-1}) \xrightarrow{E} \pi_{2n-1}(S^n) \xrightarrow{H} Z$$

that $\alpha = d[\iota_n, \iota_n] + E\beta$ for some $\beta \in \pi_{2n-2}(S^{n-1})$. Let γ be an element of $\pi_{2n-1}(S^n)$ such that $b[\alpha, \iota_n] = \alpha \circ E^{n-1}\gamma$. Then we have

$$db[[\iota_n, \iota_n], \iota_n] + b[E\beta, \iota_n] = d[\iota_n, \iota_n] \circ E^{n-1}\gamma + E(\beta \circ E^{n-2}\gamma)$$

and thus

$$[\iota_n, \iota_n] \circ (bE^n\beta - dE^{n-1}\gamma) = E(\beta \circ E^{n-2}\gamma - db\theta)$$

by use of Lemma 2 and Lemma 4. By use of Serre's decomposition of $\pi_{3n-2}(S^n)$ in odd components and by use of the isomorphism $E: \pi_{3n-3}(S^{2n-2}) \approx \pi_{3n-2}(S^{2n-1})$, we have that

$$bE^{n-1}\beta - dE^{n-2}\gamma = \delta \quad \text{and} \quad \beta \circ E^{n-2}\gamma - db\theta = \epsilon$$

for some elements δ and ϵ of the 2-components of the corresponding homotopy groups. It is computed directly that

$$d^2b\theta = b(\beta \circ E^{n-1}\beta) - \beta \circ \delta - d\epsilon.$$

Obviously $\beta \circ \delta$ and $d\epsilon$ are contained in 2-component. By the corollary of Lemma 5, $b(\beta \circ E^{n-1}\beta)$ and thus $d^2b\theta$ are contained in the 2-component. Since θ has order 3, then we have $a^2b = 4d^2b \equiv 0 \pmod 3$ and thus $ab \equiv 0 \pmod 3$.

(iv)$_0$ The case that $n=8$ and $a \equiv 0 \pmod 2$ is proved similarly to (iii).

(iv)$_1$ The case $n=8$ and $a \equiv 1 \pmod 2$. By use of the elements σ and τ of Lemma 6, any element $\alpha \in \pi_{15}(S^8)$ of $H(\alpha) = a$ is written by

$$\alpha = a\sigma + tE\tau \qquad \text{for some integer } t.$$

By use of Lemma 2, Lemma 4 and Lemma 6, we have

$$\begin{aligned}
[\alpha, \iota_8] &= a[\sigma, \iota_8] + t[E\tau, \iota_8] \\
&= [\iota_8, \iota_8] \circ (aE^7\sigma + tE^8\tau) - aE\theta \\
&= 2(a + 2t)(\sigma \circ E^7\sigma) - (a + 2t)E(\tau \circ E^6\sigma) - aE\theta.
\end{aligned}$$

Now assume that $ab \equiv 0 \pmod{24}$. By Lemma 4 and Lemma 7, we have $ab\theta = 0$ and $ab(\tau \circ E^6\sigma) = 0$. Let $t=0$ and $\gamma = 2b\sigma$, then we have $[\alpha, \iota_8] = \alpha \circ E^7\gamma$. This shows the existence of a complex of type (a, b).

Conversely, we assume the existence of a complex of type (a, b) with $a \equiv 1 \pmod 2$ and let α be the characteristic class of e^{2n} such that $H(\alpha) = a$. By Lemma 1 and Lemma 6, there exists an integer s such that

$$b[\alpha, \iota_8] = \alpha \circ sE^7\sigma.$$

It follows then from the above formula on $[\alpha, \iota_8]$ that

$$(2ab + 4bt - as)(\sigma \circ E^7\sigma) = E((ab + 2bt - st)(\tau \circ E^6\sigma) + ab\theta).$$

Concerning the direct sum decomposition, $\pi_{22}(S^8) \approx \pi_{22}(S^{15}) + \pi_{21}(S^7)$, we have the equalities

$$(2ab + 4bt - as)E^7\sigma = 0 \quad \text{and} \quad (ab + 2bt - st)(\tau \circ E^6\sigma) = -ab\theta.$$

Since $E^7\sigma$, $\tau \circ E^6\sigma$ and θ have orders 240, 8 and 3 respectively, we have that

$$2b(a + 2t) \equiv as \pmod{240}, \qquad b(a + 2t) \equiv st \pmod 8$$

and

$$ab \equiv 0 \pmod 3.$$

Let $b = 2^x b'$, $s = 2^y s'$ and $t = 2^z t'$ for some integers $x, y, z \geqq 0$ and odd integers b', s', t'. Then it follows that

$$2^{x+1} \equiv 2^y \pmod{2^4} \quad \text{and} \quad 2^x \equiv 2^{y+z} \pmod{2^3}.$$

If $x < 3$, then we have $x + 1 = y$ and $x = y + z$. But this is impossible since $z \geqq 0$. Thus we have $x \geqq 3$ and $ab = 8ab' \equiv 0 \pmod 8$.

Consequently we have $ab \equiv 0 \pmod{24}$.

Proof of Theorem 2. There exist a cell complex $L = S^n \cup e^{2n} \cup \cdots \cup e^{kn}$ and a mapping $f: L \to X$ such that $f^*: H^*(X, Z_p) \to H^*(L, Z_p)$ is an isomorphism onto. (Cf. for example Lemma 4.13 of [**14**]). Then $H^*(L, Z_p)$ is a truncated polynomial ring. Let $K = S^n \cup e^{2n} \cup e^{3n}$ be the $3n$-skeleton of L, then the type (a, b) of K satisfies $a \not\equiv 0 \pmod p$ and $b \not\equiv 0 \pmod p$. By Theorem 1, this is possible only if $n = 2$ or $n = 4$ $(p = 2$ or $3)$.

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A NOTE ON SUBGROUPS OF THE MODULAR GROUP[1]

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1. We will follow the notation of [4]. Let Γ denote the 2×2 modular group, that is, the set of all 2×2 matrices with rational integral entries and determinant 1. For each positive integer m define $\Gamma(m)$, *the principal congruence subgroup of level* m, by

$$\Gamma(m) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod{m} \right\}.$$

Let

$$T_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

and let $\Delta(m)$ be the normal subgroup of Γ generated by T_m. That is, $\Delta(m)$ is the smallest normal subgroup of Γ containing T_m. Clearly, $\Delta(m) \subset \Gamma(m)$.

In [4] Reiner considers the following questions raised in [1]:

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