A SEQUENCE OF INTEGERS RELATED TO THE BESSEL FUNCTIONS

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Let $j_{\nu,r}$ denote the zeros of $z^{-\nu}J_\nu(z)$, where $J_\nu(z)$ is the Bessel function of the first kind, and put

$$(1) \qquad \sigma_{2n}(\nu) = \sum_{r=1}^{\infty} (j_{\nu,r})^{-2n} \qquad (n = 1, 2, 3, \cdots).$$

Properties of $\sigma_{2n}(\nu)$ have been discussed in a recent paper by Kishore [2]. We remark that $\sigma_{2n}(\nu)$ is a rational function of ν with rational coefficients; the first twelve functions have been computed by Lehmer [3].

For $\nu = \pm\frac{1}{2}$, $\sigma_{2n}(\nu)$ is expressible in terms of the numbers of Bernoulli and Genocchi by means of the following formulas:

$$\sigma_{2n}(\tfrac{1}{2}) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} B_{2n},$$

$$\sigma_{2n}(-\tfrac{1}{2}) = (-1)^{n} \frac{2^{2n-2}}{(2n)!} G_{2n},$$

where

$$B^n = (B+1)^n \quad (n \neq 1), \qquad G_n = 2(1-2^n)B_n.$$

In view of the known arithmetic properties of these and related numbers, it is of some interest to look for arithmetic properties of $\sigma_{2n}(\nu)$ for other values of ν. In the present note we consider the case $\nu = 0$. Elsewhere [1] the writer has discussed the coefficients of $(J_0(\nu))^{-1}$.

It will be convenient to define

$$(2) \qquad a_r = 2^{2r} r!(r-1)!\sigma_{2r}(0) \qquad (r \geqq 1).$$

Thus the formulas [2, (14), (22)]

$$\sum_{r=1}^{n} (-1)^r 2^{2r} (r!)^2 \binom{n}{r} \binom{\nu+n}{r} \sigma_{2r}(\nu) + n = 0,$$

$$(\nu + n)\sigma_{2n}(\nu) = \sum_{r=1}^{n-1} \sigma_{2r}(\nu)\sigma_{2n-2r}(\nu)$$

reduce to

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$$\sum_{r=1}^{n}(-1)^r\binom{n}{r}\binom{n-1}{r-1}a_r+1=0 \qquad (n \geqq 1), \tag{3}$$

$$a_{n+1}=\sum_{r=1}^{n}\binom{n}{r}\binom{n}{r-1}a_r a_{n-r+1} \qquad (n \geqq 1), \tag{4}$$

respectively.

Since $a_1=1$ we find, using either (3) or (4), that

$$\begin{aligned} a_2 &= 1, \\ a_3 &= 2^2, \\ a_4 &= 3\cdot 11, \\ a_5 &= 2^3\cdot 3\cdot 19, \\ a_6 &= 2^2\cdot 5\cdot 11\cdot 43, \\ a_7 &= 2^4\cdot 3\cdot 5^2\cdot 229, \\ a_8 &= 3\cdot 5\cdot 7\cdot 167\cdot 607. \end{aligned}$$

It is evident from (4) that the a_n are positive integers. If $n=p$, a prime, it follows from (3) that

$$a_p \equiv 1 \pmod{p}. \tag{5}$$

This result can be extended. We recall that if [4]

$$\begin{aligned} n &= n_0+n_1p+n_2p^2+\cdots \qquad (0 \leqq n_j < p), \\ r &= r_0+r_1p+r_2p^2+\cdots \qquad (0 \leqq r_j < p) \end{aligned}$$

then

$$\binom{n}{r}\equiv\binom{n_0}{r_0}\binom{n_1}{r_1}\binom{n_2}{r_2}\cdots \pmod{p}. \tag{6}$$

In particular (6) implies

$$\binom{mp}{rp}\equiv\binom{m}{r}, \qquad \binom{mp-1}{rp-1}\equiv\binom{m-1}{r-1} \pmod{p}$$

and

$$\binom{mp}{r}\equiv 0 \pmod{p} \qquad (p \nmid r).$$

Thus if $n=mp$, (3) becomes

$$\sum_{r=1}^{m}(-1)^r\binom{m}{r}\binom{m-1}{r-1}a_{rp}+1\equiv 0 \pmod{p}. \tag{7}$$

Comparing (7) with (3) it is evident that

$$a_{mp} \equiv a_m \pmod{p} \tag{8}$$

and somewhat more generally

$$a_{mp^r} \equiv a_m \pmod{p} \qquad (r = 1, 2, 3, \cdots). \tag{9}$$

We show next that

$$a_n \equiv 0 \pmod{p} \qquad (p < n < 2p). \tag{10}$$

For $n=p+1$, this is obvious from (4). Assuming that (10) holds up to and including the value n, (4) implies

$$a_{n+1} \equiv \sum_{r=n-p+1}^{p} \binom{n}{r}\binom{n}{r-1} a_r a_{n-r+1} \pmod{p}. \tag{11}$$

If $n+1=p+m$, where $1 \leqq m<p$, then by (6)

$$\binom{n}{r} = \binom{p+m-1}{r} \equiv \binom{m-1}{r} \equiv 0 \pmod{p} \qquad (m \leqq r < p),$$

$$\binom{n}{p-1} = \binom{p+m-1}{p-1} \equiv \binom{m-1}{p-1} \equiv 0 \pmod{p}.$$

Thus (11) reduces to $a_{n+1} \equiv 0 \pmod{p}$. This completes the proof of (10).

It is now easy to prove the more general congruence

$$a_n \equiv 0 \pmod{p} \qquad (n > p,\ p \nmid n) \tag{12}$$

by induction. Indeed if

$$n + 1 = kp + m, \qquad 1 \leqq m < p,$$

then by (6) and the inductive hypothesis

$$a_{n+1} \equiv 2\binom{n}{m}\binom{n}{m-1} a_m a_{kp} \pmod{p};$$

but by (6)

$$\binom{n}{m} = \binom{kp+m-1}{m} \equiv \binom{m-1}{m} \equiv 0,$$

so that $a_{n+1} \equiv 0 \pmod{p}$.

We remark that

$$a_n \equiv 0 \pmod{n-1} \qquad (n > 1). \tag{13}$$

Indeed if we put

$$a_n = (n-1)b_n \qquad (n > 1),$$

(4) becomes for $n>1$

$$\begin{aligned} nb_{n+1} &= 2n(n-1)b_n + \sum_{r=2}^{n-1}\binom{n}{r}\binom{n}{r-1}(n-r)(r-1)b_r b_{n-r+1} \\ &= 2n(n-1)b_n + n^2\sum_{r=2}^{n-1}\binom{n-1}{r}\binom{n-1}{r-2}b_r b_{n-r+1}, \end{aligned}$$

so that

$$b_{n+1} = 2(n-1)b_n + n\sum_{r=2}^{n-1}\binom{n-1}{r}\binom{n-1}{r-2}b_r b_{n-r+1} \qquad (n > 1). \tag{14}$$

Since $b_2=1$ it is evident from (14) that b_n is integral for all $n \geqq 2$. This proves (13).

Returning to (4) it is clear that

$$a_{p+1} \equiv 2pa_p \pmod{p^2}.$$

Combining this with (5) we get

$$a_{p+1} \equiv 2p \pmod{p^2}. \tag{15}$$

Similarly we have for $p>2$

$$a_{p+2} \equiv 2a_{p+1} + 2\binom{p+1}{2}(p+1)a_p \pmod{p^2}.$$

Using (5) and (15) this reduces to

$$a_{p+2} \equiv 5p \pmod{p^2}. \tag{16}$$

In the same way if $p>3$ we get

$$\begin{aligned} a_{p+3} \equiv 2a_{p+2} &+ 2\binom{p+2}{2}(p+2)a_{p+1} \\ &+ 8\binom{p+2}{3}\binom{p+2}{2}a_p \pmod{p^2}, \end{aligned}$$

which reduces to

$$a_{p+3} \equiv \frac{92}{3}p \pmod{p^2} \qquad (p > 3). \tag{17}$$

To get a general result of this kind we put

$$a_{p+n} \equiv p c_n^{(p)} \pmod{p^2} \qquad (0 < n < p), \tag{18}$$

so that by (10) $c_n^{(p)}$ is integral. Now by (4)

$$\begin{aligned} a_{p+n} &= \sum_{r=1}^{p+n-1} \binom{p+n-1}{r}\binom{p+n-1}{r-1} a_r a_{p+n-r} \\ &\equiv 2\sum_{r=1}^{n} \binom{p+n-1}{r}\binom{p+n-1}{r-1} a_r a_{p+n-r} \pmod{p^2}. \end{aligned}$$

Making use of (18) and (6) this becomes

$$c_n^{(p)} \equiv 2\sum_{r=1}^{n-1} \binom{n-1}{r}\binom{n-1}{r-1} a_r c_{n-r}^{(p)} + \frac{2}{n} a_n \pmod{p} \quad (1 < n < p). \tag{19}$$

By means of (19) the $c_n^{(p)}$ can be computed. However, it is simpler to define a single sequence $\{c_n\}$ by means of

$$c_n = 2\sum_{r=1}^{n-1} \binom{n-1}{r}\binom{n-1}{r-1} a_r c_{n-r} + \frac{2}{n} a_n \qquad (n > 1) \tag{20}$$

with $c_1 = 2$. We evidently have

$$c_n^{(p)} \equiv c_n \pmod{p} \qquad (0 < n < p). \tag{21}$$

The c_n as defined by (20) are not integral. If we put

$$c_n' = n c_n \quad (n \geqq 1), \qquad c_0 = 1, \tag{22}$$

then (20) becomes

$$c_n' = 2\sum_{r=1}^{n} \binom{n}{r}\binom{n-1}{r-1} a_r' c_{n-r} \qquad (n \geqq 1); \tag{23}$$

the c_n' are therefore integral. Moreover it follows from (23) that

$$\sum_{n=1}^{\infty} \frac{c_n' \left(\frac{x}{2}\right)^{2n}}{n!(n-1)!} = 2\sum_{r=1}^{\infty} \frac{a_r \left(\frac{x}{2}\right)^{2n}}{r!(r-1)!} \sum_{n=0}^{\infty} \frac{c_n' \left(\frac{x}{2}\right)^{2n}}{n!n!}.$$

If we put

$$C(x) = \sum_{n=0}^{\infty} \frac{c_n' \left(\frac{x}{2}\right)^{2n}}{n!n!}, \qquad A(x) = \sum_{r=1}^{\infty} \frac{a_r \left(\frac{x}{2}\right)^{2n}}{r!(r-1)!}$$

then

$$\frac{x}{2}C'(x) = \sum_{n=1}^{\infty} \frac{c_n'\left(\frac{x}{2}\right)^{2n}}{n!(n-1)!},$$

so that

$$\frac{x}{2}C'(x) = 2A(x)C(x).$$

But [2]

$$A(x) = \sum_{r=1}^{\infty} \sigma_{2r}(0)x^{2r} = -\frac{1}{2}x\frac{J_0'(x)}{J_0(x)}.$$

It follows that

$$\frac{C'(x)}{C(x)} = -\frac{J_0'(x)}{J_0(x)},$$

which yields

(24) $$C(x) = (J_0(x))^{-2}.$$

We have accordingly found a simple generating function for the c_n.

It follows from (13) that if $n \equiv 1 \pmod{p^k}$ then $a_n \equiv 1 \pmod{p^k}$. Using (4) and (8) it is not difficult to show that if $n = mp^k + 1$ then

(25) $$a_n \equiv mp^k a_m \pmod{p^{k+1}}.$$

We shall now show that if

(26) $$n = n_s p^s + n_{s+1}p^{s+1} + \cdots + n_t p^t \qquad (0 \leqq n_j < p)$$

and $n_s \geqq 1$, $n_t \geqq 1$, then

(27) $$a_n \equiv 0 \pmod{p^{t-s}}.$$

The proof is by induction on n. We use (4) with n replaced by $n-1$. Let $1 \leqq r < n$ and put

$$r = r_{s'}p^{s'} + r_{s'+1}p^{s'+1} + \cdots + r_{t'}p^{t'} \qquad (0 \leqq r_j < p),$$

$$m = n - r = m_{s''}p^{s''} + m_{s''+1}p^{s''+1} + \cdots + m_{t''}p^{t''} \qquad (0 \leqq m_j < p).$$

Clearly either $s' \leqq s$ or $s'' \leqq s$; also $t' \leqq t$ and $t'' \leqq t$. If $t' = t'' = t$ there is evidently nothing to prove.

(i) $t' = t$, $t'' < t$. By the inductive hypothesis

$$a_r \equiv 0 \pmod{p^{t'-s'}}, \qquad a_{n-r} \equiv 0 \pmod{p^{t''-s''}},$$

so that

$$a_r a_{n-r} \equiv 0 \pmod{p^{t'+t''-s'-s''}}.$$

If $s' \leqq t''$ it follows (since $t'' \geqq \max(s', s'')$) that $t'+t''-s'-s'' \geqq t-s$. If however $s' > t''$ we examine the binomial coefficient $C_{n-1,n}$.

We recall that if

$$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqq n_j < p),$$
$$S(n) = n_0 + n_1 + \cdots + n_k,$$

then $n!$ is divisible by exactly p^e, where

$$(p-1)^e = n - S(n).$$

It follows that $\binom{n-1}{r}$ is divisible by $p^{s'-s''}$. Since $(t'+t''-s'-s'')+(s'-s'')=t+t''-2s \geqq t-s$ we get

$$\binom{n-1}{r} a_r a_{n-r} \equiv 0 \pmod{p^{t-s}}.$$

(ii) $t' < t$, $t'' < t$. We may suppose that $t'' \leqq t' = t-1$. Also it is clear that $s' \leqq t''$. Then

$$t' + t'' - s' - s'' \geqq t - s - 1;$$

indeed if s min (s', s'') we get

$$t' + t'' - s' - s'' \geqq t - s.$$

Thus only the case $s = \min(s', s'')$ requires further examination. With the present hypothesis we evidently have

$$\binom{n}{r} \equiv 0 \pmod{p};$$

but when $s = \min(s', s'')$ then either n and r or n and $n-r$ are divisible by the same power of p. It follows that either

$$\binom{n-1}{r} \quad \text{or} \quad \binom{n-1}{r-1}$$

is divisible by p. Consequently

$$\binom{n-1}{r}\binom{n-1}{r-1} a_r a_{n-r} \equiv 0 \pmod{p^{t-s}}.$$

This completes the proof of (26).

SUMMARY. The sequence of positive integers $\{a_n\}$ defined by (2) —or alternatively by (3) or (4)—have the following properties.

1. $a_{mp} \equiv a_m, \quad a_p \equiv 1 \pmod{p}$.
2. $a_n \equiv 0 \pmod{p} \quad (n > p,\ p \nmid n)$.
3. $a_n \equiv 0 \pmod{n-1} \quad (n > 1)$.
4. $a_{p+n} \equiv c_n p \pmod{p^2} \quad (1 \leqq n < p)$,

where the c_n are defined by

$$1 + \sum_{n=1}^{\infty} \frac{c_n \left(\frac{x}{2}\right)^{2n}}{(n-1)!(n-1)!} = (J_0(x))^{-2};$$

moreover nc_n is integral.

5. If $n = mp^k + 1$ then

$$a_n \equiv mp^k a_m \pmod{p^{k+1}}.$$

6. If $p^s \mid n$, $p^{s+1} \nmid n$, $p^t \leqq n < p^{t+1}$ then

$$a_n \equiv 0 \pmod{p^{t-s}}.$$

The following values of a_n were computed by R. Carlitz in the Duke University Computing Laboratory.

$$a_9 = 2^4 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 1607$$
$$a_{10} = 2^3 \cdot 3^2 \cdot 7 \cdot 199 \cdot 328981$$
$$a_{11} = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 83 \cdot 3000553$$
$$a_{12} = 2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 2389 \cdot 4153159$$
$$a_{13} = 2^5 \cdot 3^2 \cdot 5 \cdot 7^4 \cdot 11 \cdot 29 \cdot 97 \cdot 139 \cdot 1663$$
$$a_{14} = 2^4 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 37 \cdot 107 \cdot 1283 \cdot 5952613$$
$$a_{15} = 2^6 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 563 \cdot 797 \cdot 227966279$$
$$a_{16} = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 41 \cdot 23907005144172 53$$
$$a_{17} = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 461 \cdot 1342361 \cdot 33327739$$
$$a_{18} = 2^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 79 \cdot 199729 \cdot 139135943558279$$
$$a_{19} = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot a_{19}'$$
$$a_{20} = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 1109 \cdot a_{20}'$$
$$a_{21} = 2^6 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 47 \cdot a_{21}'$$
$$a_{22} = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot a_{22}'$$
$$a_{23} = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot a_{23}'$$
$$a_{24} = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a_{24}'$$

$$a_{25} = 2^6\cdot 3^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot a'_{25}$$
$$a_{26} = 2^5\cdot 3^2\cdot 5^2\cdot 7\cdot 11\cdot 17\cdot 19\cdot 23^2\cdot 79\cdot a'_{26}$$
$$a_{27} = 2^7\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 43^3\cdot a'_{27}$$
$$a_{28} = 2^4\cdot 3^3\cdot 5^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot a'_{28}$$
$$a_{29} = 2^7\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot a'_{29}$$
$$a_{30} = 2^6\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot a'_{30}$$

The numbers $a'_{19}, \cdots, a'_{30}$ have not been factored completely but at any rate have no prime divisors $<10^4$. The number a'_{30} has 47 digits.

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