ON THE LOGARITHMIC MEANS OF THE SUCCESSIVELY DERIVED CONJUGATE SERIES OF A FOURIER SERIES

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1. Definition [6]. A series \( \sum c_n \) is said to be summable \((R, \log \omega, k)\) or summable \((R, k)\), to sum \( s\), if

\[
R_k(\omega) = \frac{1}{(\log \omega)^k} \sum_{n<\omega} \left( \frac{\log \omega}{n} \right)^k c_n
\]

tends to a limit \( s\), as \( \omega \to \infty \). \( R_k(\omega) \) is called the \((R, \log \omega, k)\) mean of \( \sum c_n \).

Let the Fourier series of a function \( f(\theta) \in L \) over \(( -\pi, \pi)\) and periodic outside this range with period \( 2\pi \), be

\[
(1.1) \quad f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(\theta).
\]

The series conjugate to the above Fourier series is

\[
(1.2) \quad \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_{n=1}^{\infty} B_n(\theta),
\]

and the \( r \)th derived series of the above conjugate series is

\[
(1.3) \quad \sum_{n=1}^{\infty} \frac{d^r}{d\theta^r} (b_n \cos n\theta - a_n \sin n\theta).
\]

Suppose that \( P(t) \) is a polynomial of \((r-1)\)th degree in \( t \), such that

\[
h(t) = \frac{1}{2^r} \left[ \{ f(x + t) - P(t) \} - (-1)^r \{ f(x - t) - P(-t) \} \right]
\]

is integrable \((L)\) over \(( -\pi, \pi)\) and is defined by periodicity outside this range with period \( 2\pi \).

In what follows we shall write

\[
\psi(t) = \frac{1}{2} \{ f(x + t) - f(x - t) \},
\]

\[
\psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - u)^{\alpha-1} \psi(u) du, \quad \alpha > 0,
\]

\[
\psi_\alpha(t) = \frac{\Gamma(\alpha + 1)}{t^\alpha} \psi_\alpha(t),
\]

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\[
\overline{\psi_a(t)} = \frac{1}{\Gamma(\alpha)} \int_t^\infty \left( \log \frac{u}{t} \right)^{\alpha-1} \frac{\psi(u)}{u} \, du, \quad \alpha > 0,
\]

\[
\check{\psi}_a(t) = \Gamma(\alpha + 1) \left( \log \frac{1}{t} \right)^{-\alpha} \overline{\psi_a(t)},
\]

and define \(\theta_a(t), \check{\theta}_a(t), \overline{\theta}_a(t), \overline{\theta}_a(t)\) and \(\overline{h_a(t)}\) in a similar way.

Also we shall denote by \(R^0_\alpha(\omega)\) and \(R^\delta_\alpha(\omega)\) the \((R, \delta)\) means of (1.2) and (1.3), respectively and by \(B^\delta(\omega)\) the \((R, \delta)\) mean of the conjugate series of \(h(t)\).

2. It is known that if \(\theta_a(t)\to 0\) as \(t\to 0\), for \(\alpha \geq 0\), then

\begin{equation}
C_a(n) \sim \frac{2l}{\pi} \log n,
\end{equation}

where \(C_a(n)\) denotes the \((C, \alpha)\) mean of the conjugate series (1.2) at \(\theta = x\). This result for \(\alpha = 0\) is the classical result of Lukàcs [2] and for \(\alpha > 0\) it has been obtained by Obrechkoff [5].

Considering logarithmic means instead of Cesàro means, we find that if for \(\rho > -1, \alpha \geq 0\),

\[
\int_t^\infty \left| \frac{\overline{\theta}_a(u)}{u} \right| \, du = o\left( \left( \log \frac{1}{t} \right)^{\rho+1} \right), \quad \text{as } t \to 0,
\]

then

\[
R_\alpha^\delta(\omega) = \frac{2l}{\pi(\alpha + 1)} \log \omega + o\left( (\log \omega)^{\rho+1} \right), \quad \text{as } \omega \to \infty.
\]

(See Lemma 2 of the present paper.) For \(\rho = 0\) this gives a result analogous to (2.1), which, although not explicitly mentioned, may also be obtained from a result of Misra [3].

For the first derived series of the conjugate series, Mohanty and Nanda [4] have recently proved the following theorem:

**Theorem A.** If

\[
\int_t^\infty \left| \{ f(x + u) + f(x - u) - 2f(x) \} \left( \frac{4 \sin \frac{u}{2} }{2} \right) - l \right| u^{-1} \, du
\]

\[
= o \left( \log \frac{1}{t} \right), \quad \text{as } t \to 0,
\]

then \(\lim_{n \to \infty} (\sigma_{2n} - \sigma_n) = l/\pi \log 2\), where \(\sigma_n\) denotes the \((R, \log n, 1)\) mean of the first derived conjugate series, at \(\theta = x\).
The object of the present paper is to generalize this theorem to the case of the $r$th derived series of the conjugate series and to obtain a more refined result by replacing $\sigma_n$ and $\sigma_n$ by $R'_r(\lambda \omega)$ and $R'_r(\omega)$ respectively and also by replacing the condition of the theorem by a set of weaker conditions. Thus we prove the following theorem which gives a more far-reaching result.

**Theorem 1.** If

$$\int_t^x | h(u) - 1 | u^{-1} du = o\left\{ \left( \log \frac{1}{t} \right)^r \right\}, \text{ as } t \to 0, \text{ for } r \geq 1,$$

and

$$\int_t^x | h_{r-1}(u) - s | u^{-1} du = o \left( \log \frac{1}{t} \right), \text{ as } t \to 0, \text{ for } r \geq 1,$$

then

$$[R'_r(\lambda \omega) - R'_r(\omega)] \sim \frac{2^r}{\pi} \frac{r!}{r + 1} \log \lambda, \quad \lambda > 1.$$ 

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3. We require the following lemmas for the proof of our theorem:

**Lemma 1** [8].

If

$$M_p(t) = \int_0^1 \left( \log \frac{1}{u} \right)^p \sin ut \, du, \quad p > -1,$$

then

$$\frac{d}{dt} \left\{ t M_p(t) \right\} = p M_{p-1}(t), \quad p > 0.$$ 

**Lemma 2.** If for $p > -1$, $\alpha \geq 0$,

$$\int_t^x \left| \frac{\theta_\alpha(u)}{u} \right| u^{\alpha} \, du = o \left\{ \left( \log \frac{1}{t} \right)^{\alpha+1} \right\}, \text{ as } t \to 0,$$

then

$$R_\alpha^2(\omega) = \frac{2}{\pi(\alpha + 1)} \log \omega + o\left\{ (\log \omega)^{\alpha+1} \right\}, \text{ as } \omega \to \infty.$$ 

**Proof of Lemma 2.** Suppose that $h = [\alpha]$. We have by well-known arguments
\[ R(\omega) = \sum_{n<\omega} B_n(x) = \frac{1}{\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} (1 - \cos \omega t)dt + o(1) \]

\[ = \frac{1}{\pi} \int_0^\pi \theta(t) \cot \frac{t}{2} (1 - \cos \omega t)dt + \frac{l}{\pi} \int_0^\pi \cot \frac{t}{2} (1 - \cos \omega t)dt + o(1) \]

\[ = \frac{1}{\pi} \int_0^\pi \theta(t) \left( \cot \frac{t}{2} - \frac{2}{t} \right) (1 - \cos \omega t)dt + \frac{2l}{\pi} \log \omega + O(1) \]

\[ = \frac{2\omega}{\pi} \int_0^\pi \Theta_1(t) \sin \omega t dt + \frac{2l}{\pi} \log \omega + O(1). \]

Considering first the case when \( \alpha \) is nonintegral, we have

\[ R_\alpha(\omega) = \frac{1}{(\log \omega)^\alpha} \sum_{n<\omega} \left( \log \frac{\omega}{n} \right)^\alpha B_n = \frac{\alpha}{(\log \omega)^\alpha} \int_1^\omega \left( \frac{\log x}{x} \right)^{\alpha-1} R(x) x dx \]

\[ = \frac{2\omega}{\pi (\log \omega)^\alpha} \int_1^\omega \left( \frac{\log x}{x} \right)^{\alpha-1} \log x \frac{dx}{x} \]

\[ + \frac{2\alpha}{\pi (\log \omega)^\alpha} \int_1^\omega \left( \frac{\log x}{x} \right)^{\alpha-1} dx \int_0^\pi \Theta_1(t) \sin xtdt \]

\[ + O(1) \]

\[ = \frac{2\omega}{\pi} \log \omega \alpha + \frac{2\alpha}{\pi (\log \omega)^\alpha} \int_0^\pi \Theta_1(t)dt \int_1^\omega \left( \log \frac{\omega}{x} \right)^{\alpha-1} \sin xtdt \]

\[ + O(1) \]

\[ = \frac{2\omega}{\pi} \log \omega \alpha + \frac{2\alpha}{\pi (\log \omega)^\alpha} \int_0^\pi \Theta_1(t) M_{\alpha-1}(\omega t)dt \]

\[ - \frac{2\alpha \omega}{\pi (\log \omega)^\alpha} \int_0^\pi \Theta_1(t) \int_0^{1/\omega} \left( \log \frac{1}{v} \right)^{\alpha-1} \sin (\omega tv)dv + O(1) \]

\[ = \frac{2\omega}{\pi} \log \omega \alpha + \frac{2\alpha \omega}{\pi (\log \omega)^\alpha} \int_0^\pi \Theta_1(t) t M_{\alpha-1}(\omega t)dt + O(1); \]

where
Thus, integrating by parts,

$$M_2(t) = \int_0^1 \left( \log \frac{1}{u} \right)^p \sin \omega t \, du, \quad p > -1.$$  

$$R_2(\omega) = \frac{2I}{\pi} \frac{\log \omega}{\alpha + 1} + \frac{c_1}{(\log \omega)^a} \left[ \int_0^\infty \bar{\Theta}_{\nu+1}(t) M_{\nu-1}(\omega t) \, dt + O(1) \right]$$

$$= \frac{2I}{\pi} \frac{\log \omega}{\alpha + 1} + \frac{c_1}{(\log \omega)^a} \int_0^\infty \bar{\Theta}_{\nu}(u) \left\{ \int_0^u \left( \log \frac{u}{t} \right)^{\nu-\alpha} \frac{\Theta_{\nu}(u)}{u} \, dt \right\} \, du + O(1) \tag{3.1}$$

$$= \frac{2I}{\pi} \frac{\log \omega}{\alpha + 1} + \frac{c_1}{(\log \omega)^a} \left[ J_1 - J_2 \right] + O(1), \text{ say.}$$

Now

$$J_1 = O \left[ \int_0^{1/\omega} \left( \frac{\Theta_{\nu}(u)}{u} \right)^2 \, du \right]$$

$$= O \left[ \omega \int_0^{1/\omega} \left( \frac{\Theta_{\nu}(u)}{u} \right) \, du \right]$$

$$= O \left[ \omega \left\{ -u \int_u^1 \left( \frac{\Theta_{\nu}(u)}{u} \right) \, du \right\} \right]$$

$$= o((\log \omega)^{a+\epsilon+1}).$$

Again
(3.3) \[ J_2 = O \left[ \int_{1/\omega}^{\omega} \frac{|\hat{\Theta}_a(u)|}{u} \, du \right] = o \{ (\log \omega)^{\alpha+\sigma+1} \}, \]

and the result follows from (3.1), (3.2) and (3.3).

For the case when \( \alpha \) is an integer, the proof becomes simpler.

**Lemma 3** [3]. If

\[ \int_{t}^{\infty} \hat{\Theta}_a(u) | u^{-1} \, du = o \left\{ \left( \log \frac{1}{t} \right)^{\alpha+1} \right\}, \] as \( t \to 0 \), for \( \alpha \geq 0, \)

then

\[ \lim_{\omega \to \infty} \left[ R^{\sigma+1}_a(\lambda \omega) - R^{\sigma+1}_0(\omega) \right] = \frac{2l}{\pi(\alpha+2)} \log \lambda, \quad \lambda > 1. \]

**Lemma 4** [1].

\[ R^{\sigma}_r(\omega) = (-1)^r \left[ a_0 B^r(\omega) + a_1 \frac{B^{r-1}(\omega)}{\log \omega} + \cdots + a_r \frac{B^0(\omega)}{(\log \omega)^r} \right], \]

where \( a_0 = (-1)^r r! \) and \( a_p \)'s for \( p = 1, 2, \ldots, r \), are constants independent of \( \omega \).

**Lemma 5** [7]. Let \( V(\omega) \) and \( W(\omega) \) be two positive nondecreasing functions of \( \omega \). Let

\[ U^r_r(\omega) = \{ V(\omega) \}^{1-p/k} \{ W(\omega) \}^{p/k}. \]

Then (i) \( h^r(\omega) = o \{ W(\omega) \} \) and (ii) \( h(\omega) = o \{ V(\omega) \} \) together imply that

\[ h^p(\omega) = o \{ U^p(\omega) \} \] for \( 0 < p < k \), where \( h^p(\omega) \) denotes the \((R, p)\) mean of \( h(\omega) \). Also the result remains true if in any one of the conditions (i) and (ii), "o" is replaced by "O".

4. Proof of Theorem 1. By Lemma 4 we have

\[ R^{\sigma}_r(\omega) = r! B^r(\omega) + (-1)^r \left[ a_1 \frac{B^{r-1}(\omega)}{\log \omega} + \cdots + a_r \frac{B^0(\omega)}{(\log \omega)^r} \right]. \]

Also by Lemma 2 and by the first condition of the theorem

\[ B^0(\omega) = o \{ (\log \omega)^r \}, \quad \text{for } r > 1, \]

and by the second condition

\[ B^{r-1}(\omega) = \frac{2l}{\pi r} \log \omega + o(\log \omega) \]

\[ = O(\log \omega). \]
Now by Lemma 5 we have

\[ B^1(\omega) = o\left( (\log \omega)^{-1} \right), \]
\[ B^2(\omega) = o\left( (\log \omega)^{-2} \right), \]
\[ \ldots \ldots \ldots \ldots \]
\[ B^{-2}(\omega) = o\left( (\log \omega)^2 \right). \]

Thus, supposing that \( r > 1 \),

\[
R(\lambda) - R(\omega) = r!\left[ B'(\lambda) - B'(\omega) \right] + (-1)^r \alpha \left\{ \frac{B^{r-1}(\lambda)}{\log (\lambda)} - \frac{B^{r-1}(\omega)}{\log (\omega)} \right\} \\
+ \ldots + (-1)^r \alpha \left\{ \frac{B^0(\lambda)}{(\log (\lambda))^r} - \frac{B^0(\omega)}{(\log (\omega))^r} \right\} \\
= r!\left[ B'(\lambda) - B'(\omega) \right] + (-1)^r \alpha \frac{2\lambda}{\pi \lambda} \left\{ \frac{\log (\lambda)}{\log (\omega)} \right\} \\
+ o\left( \frac{\log (\lambda)}{\log (\omega)} \right) + o\left( \frac{\log (\omega)}{\log (\omega)} \right) + \ldots + o\left( \frac{\log (\lambda)}{(\log (\omega))^r} \right) + o\left( \frac{(\log (\omega))^r}{(\log (\omega))^r} \right) \\
= r!\left[ B'(\lambda) - B'(\omega) \right] + o(1) = r! \frac{2\lambda}{\pi \lambda} + o(1)
\]

by Lemma 3. This completes the proof.

References


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