ON THE LOGARITHMIC MEANS OF THE SUCCESSIVELY DERIVED CONJUGATE SERIES OF A FOURIER SERIES

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1. **Definition** [6]. A series $\sum c_n$ is said to be summable $(R, \log \omega, k)$ or summable (R, k), to sum s, if

$$R_k(\omega) = \frac{1}{(\log \omega)^k} \sum_{n<\omega} \left(\log \frac{\omega}{n}\right)^k c_n$$

tends to a limit s, as $\omega \to \infty$. $R_k(\omega)$ is called the $(R, \log \omega, k)$ mean of $\sum c_n$.

Let the Fourier series of a function $f(\theta) \in L$ over $(-\pi, \pi)$ and periodic outside this range with period 2π, be

$$(1.1) \quad f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(\theta).$$

The series conjugate to the above Fourier series is

$$(1.2) \qquad \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_{n=1}^{\infty} B_n(\theta),$$

and the rth derived series of the above conjugate series is

$$(1.3) \qquad \sum_{n=1}^{\infty} \frac{d^r}{d\theta^r} (b_n \cos n\theta - a_n \sin n\theta).$$

Suppose that $P(t)$ is a polynomial of $(r-1)$th degree in t, such that

$$h(t) = \frac{1}{2t^r} [\{f(x+t) - P(t)\} - (-1)^r \{f(x-t) - P(-t)\}]$$

is integrable (L) over $(-\pi, \pi)$ and is defined by periodicity outside this range with period 2π.

In what follows we shall write

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \psi(u) du, \qquad \alpha > 0,$$

$$\psi_\alpha(t) = \frac{\Gamma(\alpha+1)}{t^\alpha} \Psi_\alpha(t),$$

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$$\overline{\Psi}_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_t^\pi \left(\log\frac{u}{t}\right)^{\alpha-1}\frac{\psi(u)}{u}\,du, \qquad \alpha > 0,$$

$$\bar{\psi}_\alpha(t) = \Gamma(\alpha+1)\left(\log\frac{1}{t}\right)^{-\alpha}\overline{\Psi}_\alpha(t),$$

and define $\theta_\alpha(t)$, $\overline{\Theta}_\alpha(t)$, $\bar{\theta}_\alpha(t)$, $\overline{H}_\alpha(t)$ and $\bar{h}_\alpha(t)$ in a similar way.

Also we shall denote by $R_0^\delta(\omega)$ and $R_r^\delta(\omega)$ the (R, δ) means of (1.2) and (1.3), respectively and by $B^\delta(\omega)$ the (R, δ) mean of the conjugate series of $h(t)$.

2. It is known that if $\theta_\alpha(t)\to 0$ as $t\to 0$, for $\alpha \geqq 0$, then

$$C_\alpha(n) \sim \frac{2l}{\pi}\log n, \tag{2.1}$$

where $C_\alpha(n)$ denotes the (C, α) mean of the conjugate series (1.2) at $\theta = x$. This result for $\alpha = 0$ is the classical result of Lukàcs [2] and for $\alpha > 0$ it has been obtained by Obrechkoff [5].

Considering logarithmic means instead of Cesàro means, we find that if for $p > -1$, $\alpha \geqq 0$,

$$\int_t^\pi \frac{|\bar{\theta}_\alpha(u)|}{u}\,du = o\left\{\left(\log\frac{1}{t}\right)^{p+1}\right\}, \text{ as } t\to 0,$$

then

$$R_0^\alpha(\omega) = \frac{2l}{\pi(\alpha+1)}\log\omega + o\{(\log\omega)^{p+1}\}, \text{ as } \omega\to\infty.$$

(See Lemma 2 of the present paper.) For $p = 0$ this gives a result analogous to (2.1), which, although not explicitly mentioned, may also be obtained from a result of Misra [3].

For the first derived series of the conjugate series, Mohanty and Nanda [4] have recently proved the following theorem:

THEOREM A. *If*

$$\int_t^\pi \left|\{f(x+u) + f(x-u) - 2f(x)\} \Big/ \left(4\sin\frac{u}{2}\right) - l\right| u^{-1}du = o\left(\log\frac{1}{t}\right), \text{ as } t\to 0,$$

then $\lim_{n\to\infty}(\sigma_{2n} - \sigma_n) = l/\pi \log 2$, *where* σ_n *denotes the* $(R, \log n, 1)$ *mean of the first derived conjugate series, at* $\theta = x$.

The object of the present paper is to generalize this theorem to the case of the rth derived series of the conjugate series and to obtain a more refined result by replacing σ_{2n} and σ_n by $R_r^r(\lambda\omega)$ and $R_r^r(\omega)$ respectively and also by replacing the condition of the theorem by a set of weaker conditions. Thus we prove the following theorem which gives a more far-reaching result.

THEOREM 1. *If*

$$\int_t^\pi |h(u) - l| u^{-1} du = o\left\{\left(\log \frac{1}{t}\right)^r\right\}, \quad as\ t \to 0,\ for\ r \geqq 1,$$

and

$$\int_t^\pi |\bar{h}_{r-1}(u) - s| u^{-1} du = o\left(\log \frac{1}{t}\right), \quad as\ t \to 0,\ for\ r \geqq 1,$$

then

$$[R_r^r(\lambda\omega) - R_r^r(\omega)] \sim \frac{2l}{\pi} \frac{r!}{r+1} \log \lambda, \qquad \lambda > 1.$$

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3. We require the following lemmas for the proof of our theorem:

LEMMA 1 [8]. *If*

$$M_p(t) = \int_0^1 \left(\log \frac{1}{u}\right)^p \sin ut\, du, \qquad p > -1,$$

then

$$\frac{d}{dt}\{tM_p(t)\} = pM_{p-1}(t), \qquad p > 0.$$

LEMMA 2. *If for* $p > -1$, $\alpha \geqq 0$,

$$\int_t^\pi \frac{|\bar{\theta}_\alpha(u)|}{u} du = o\left\{\left(\log \frac{1}{t}\right)^{p+1}\right\}, \quad as\ t \to 0,$$

then

$$R_0^\alpha(\omega) = \frac{2l}{\pi(\alpha+1)} \log \omega + o\{(\log \omega)^{p+1}\}, \quad as\ \omega \to \infty.$$

PROOF OF LEMMA 2. Suppose that $h = [\alpha]$. We have by well-known arguments

$$
\begin{aligned}
R(\omega) &= \sum_{n<\omega} B_n(x) = \frac{1}{\pi}\int_0^\pi \psi(t)\cot\frac{t}{2}(1-\cos\omega t)dt + o(1)\\
&= \frac{1}{\pi}\int_0^\pi \theta(t)\cot\frac{t}{2}(1-\cos\omega t)dt\\
&\quad + \frac{l}{\pi}\int_0^\pi \cot\frac{t}{2}(1-\cos\omega t)dt + o(1)\\
&= \frac{1}{\pi}\int_0^\pi \theta(t)\left(\cot\frac{t}{2}-\frac{2}{t}\right)(1-\cos\omega t)dt\\
&\quad + \frac{2}{\pi}\int_0^\pi \theta(t)\frac{1-\cos\omega t}{t}dt + \frac{2l}{\pi}\log\omega + O(1)\\
&= \frac{2}{\pi}\int_0^\pi \theta(t)\frac{1-\cos\omega t}{t}dt + \frac{2l}{\pi}\log\omega + O(1).\\
&= \frac{2\omega}{\pi}\int_0^\pi \overline{\Theta}_1(t)\sin\omega t\,dt + \frac{2l}{\pi}\log\omega + O(1).
\end{aligned}
$$

Considering first the case when α is nonintegral, we have

$$
\begin{aligned}
R_0^\alpha(\omega) &= \frac{1}{(\log\omega)^\alpha}\sum_{n<\omega}\left(\log\frac{\omega}{n}\right)^\alpha B_n = \frac{\alpha}{(\log\omega)^\alpha}\int_1^\omega\left(\log\frac{\omega}{x}\right)^{\alpha-1}\frac{R(x)}{x}dx\\
&= \frac{2l}{\pi}\frac{\alpha}{(\log\omega)^\alpha}\int_1^\omega\left(\log\frac{\omega}{x}\right)^{\alpha-1}\frac{\log x}{x}dx\\
&\quad + \frac{2\alpha}{\pi}\frac{1}{(\log\omega)^\alpha}\int_1^\omega\left(\log\frac{\omega}{x}\right)^{\alpha-1}dx\int_0^\pi\overline{\Theta}_1(t)\sin xt\,dt + O(1)\\
&= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + \frac{2\alpha}{\pi(\log\omega)^\alpha}\int_0^\pi\overline{\Theta}_1(t)dt\int_1^\omega\left(\log\frac{\omega}{x}\right)^{\alpha-1}\sin xt\,dt\\
&\quad + O(1)\\
&= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + \frac{2\alpha\omega}{\pi(\log\omega)^\alpha}\int_0^\pi\overline{\Theta}_1(t)M_{\alpha-1}(\omega t)dt\\
&\quad - \frac{2\alpha\omega}{\pi(\log\omega)^\alpha}\int_0^\pi\overline{\Theta}_1(t)\int_0^{1/\omega}\left(\log\frac{1}{v}\right)^{\alpha-1}\sin(\omega tv)dv + O(1)\\
&= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + \frac{2\alpha\omega}{\pi(\log\omega)^\alpha}\int_0^\pi\frac{\overline{\Theta}_1(t)}{t}tM_{\alpha-1}(\omega t)dt + O(1);
\end{aligned}
$$

where

$$M_p(t) = \int_0^1 \left(\log \frac{1}{u}\right)^p \sin ut\, du, \qquad p > -1.$$

Thus, integrating by parts,

$$\begin{aligned}
R_0^\alpha(\omega) &= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + c\,\frac{\omega}{(\log\omega)^\alpha}\int_0^\pi \overline{\Theta}_{h+1}(t) M_{\alpha-h-1}(\omega t)\,dt + O(1) \\
&= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + c_1\,\frac{\omega}{(\log\omega)^\alpha}\int_0^\pi M_{\alpha-h-1}(\omega t) \\
&\qquad\qquad \cdot\left\{\int_t^\pi \left(\log\frac{u}{t}\right)^{h-\alpha}\frac{\overline{\Theta}_\alpha(u)}{u}\,du\right\}dt + O(1) \\
&= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + \frac{c_1\omega}{(\log\omega)^\alpha}\int_0^\pi \frac{\overline{\Theta}_\alpha(u)}{u} \\
&\qquad\qquad \cdot\left\{\int_0^u \left(\log\frac{u}{t}\right)^{h-\alpha} M_{\alpha-h-1}(\omega t)\,dt\right\}du + O(1) \qquad (3.1) \\
&= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + \frac{c_1}{(\log\omega)^\alpha} \\
&\qquad\qquad \cdot\left[\left\{\int_0^{1/\omega} + \int_{1/\omega}^\pi\right\}\frac{\overline{\Theta}_\alpha(u)}{u}(1-\cos\omega u)\,du\right] + O(1) \\
&= \frac{2l}{\pi}\frac{\log\omega}{\alpha+1} + \frac{c_1}{(\log\omega)^\alpha}\,[J_1 - J_2] + O(1), \text{ say.}
\end{aligned}$$

Now

$$\begin{aligned}
J_1 &= O\left[\int_0^{1/\omega} \frac{|\overline{\Theta}_\alpha(u)|}{u}\,\omega^2 u^2 du\right] \\
&= O\left[\omega\int_0^{1/\omega} \frac{|\overline{\Theta}_\alpha(u)|}{u}\cdot u\,du\right] \\
&= O\left[\omega\left\{-u\int_u^\pi \frac{|\overline{\Theta}_\alpha(u)|}{u}\,du\right\}_{u=0}^{u=1/\omega}\right. \qquad (3.2) \\
&\qquad\left. + \omega\left\{\int_0^{1/\omega}\int_u^\pi \frac{|\overline{\Theta}_\alpha(u)|}{u}\,du\right\}\right] \\
&= o[(\log\omega)^{\alpha+p+1}].
\end{aligned}$$

Again

$$(3.3)\qquad J_2 = O\left[\int_{1/\omega}^{\pi} \frac{|\bar{\Theta}_\alpha(u)|}{u}\,du\right] = o\{(\log\omega)^{\alpha+p+1}\},$$

and the result follows from (3.1), (3.2) and (3.3).

For the case when α is an integer, the proof becomes simpler.

LEMMA 3 [3]. *If*

$$\int_t^{\pi} |\overline{\Theta}_\alpha(u)|\, u^{-1}du = o\left\{\left(\log\frac{1}{t}\right)^{\alpha+1}\right\},\ \textit{as } t\to 0, \textit{ for } \alpha \geqq 0,$$

then

$$\lim_{\omega\to\infty}\left[R_0^{\alpha+1}(\lambda\omega) - R_0^{\alpha+1}(\omega)\right] = \frac{2l}{\pi(\alpha+2)}\log\lambda, \qquad \lambda > 1.$$

LEMMA 4 [1].

$$R_r^r(\omega) = (-1)^r\left[\alpha_0 B^r(\omega) + \alpha_1\frac{B^{r-1}(\omega)}{\log\omega} + \cdots + \alpha_r\frac{B^0(\omega)}{(\log\omega)^r}\right],$$

where $\alpha_0 = (-1)^r r!$ *and* α_p*'s for* $p = 1, 2, \cdots, r$, *are constants independent of* ω.

LEMMA 5 [7]. *Let* $V(\omega)$ *and* $W(\omega)$ *be two positive nondecreasing functions of* ω. *Let*

$$U_r(\omega) = \{V(\omega)\}^{1-p/k}\{W(\omega)\}^{p/k}.$$

Then (i) $h^k(\omega) = o\{W(\omega)\}$ *and* (ii) $h(\omega) = o\{V(\omega)\}$ *together imply that* $h^p(\omega) = o\{U_p(\omega)\}$ *for* $o < p < k$, *where* $h^p(\omega)$ *denotes the* (R, p) *mean of* $h(\omega)$. *Also the result remains true if in any one of the conditions* (i) *and* (ii), *"o" is replaced by "O".*

4. Proof of Theorem 1. By Lemma 4 we have

$$R_r^r(\omega) = r!B^r(\omega) + (-1)^r\left[\alpha_1\frac{B^{r-1}(\omega)}{\log\omega} + \cdots + \alpha_r\frac{B^0(\omega)}{(\log\omega)^r}\right].$$

Also by Lemma 2 and by the first condition of the theorem

$$B^0(\omega) = o\{(\log\omega)^r\}, \qquad \text{for } r > 1,$$

and by the second condition

$$\begin{aligned} B^{r-1}(\omega) &= \frac{2l}{\pi r}\log\omega + o(\log\omega) \\ &= O(\log\omega). \end{aligned}$$

Now by Lemma 5 we have

$$\begin{aligned} B^1(\omega) &= o\{(\log \omega)^{r-1}\}, \\ B^2(\omega) &= o\{(\log \omega)^{r-2}\}, \\ &\cdots\cdots\cdots, \\ B^{r-2}(\omega) &= o\{(\log \omega)^2\}. \end{aligned}$$

Thus, supposing that $r > 1$,

$$\begin{aligned} &R_r^r(\lambda\omega) - R_r^r(\omega) \\ &= r!\{B^r(\lambda\omega) - B^r(\omega)\} + (-1)^r\alpha_1\left\{\frac{B^{r-1}(\lambda\omega)}{\log(\lambda\omega)} - \frac{B^{r-1}(\omega)}{\log\omega}\right\} \\ &\quad + \cdots + (-1)^r\alpha_r\left\{\frac{B^0(\lambda\omega)}{\{\log(\lambda\omega)\}^r} - \frac{B^0(\omega)}{(\log\omega)^r}\right\} \\ &= r!\{B^r(\lambda\omega) - B^r(\omega)\} + (-1)^r\frac{\alpha_1 2l}{\pi r}\left\{\frac{\log(\lambda\omega)}{\log(\lambda\omega)} - \frac{\log\omega}{\log\omega}\right\} \\ &\quad + o\left\{\frac{\log(\lambda\omega)}{\log(\lambda\omega)}\right\} + o\left\{\frac{\log\omega}{\log\omega}\right\} + \cdots + o\left\{\frac{\{\log(\lambda\omega)\}^r}{\{\log(\lambda\omega)\}^r}\right\} + o\left\{\frac{(\log\omega)^r}{(\log\omega)^r}\right\} \\ &= r!\{B^r(\lambda\omega) - B^r(\omega)\} + o(1) = r!\frac{2l}{\pi}\frac{\log\lambda}{r+1} + o(1) \end{aligned}$$

by Lemma 3. This completes the proof.

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