ON THE BEHAVIOUR OF THE FOURIER COEFFICIENTS

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1.1. Let $f(\theta)$ be a function integrable in the sense of Lebesgue over $(-\pi, \pi)$ and periodic outside this range with period 2π. Let the Fourier series associated with $f(\theta)$ be

$$(1.1.1) \quad f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(\theta).$$

Then the series conjugate to the above Fourier series is

$$(1.1.2) \qquad \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_{n=1}^{\infty} B_n(\theta).$$

We write

$$\begin{aligned}
\phi(t) &= \phi_0(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\}; \\
\psi(t) &= \tfrac{1}{2}\{f(x+t) - f(x-t)\}; \\
\theta(t) &= \psi(t) - l; \\
\Phi_\beta(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1}\phi(u)\,du, && \beta > 0; \\
\Phi_\beta(t) &= \frac{d}{dt}\,\Phi_{\beta+1}(t), && -1 < \beta < 0; \\
\phi_\beta(t) &= \Gamma(\beta+1)t^{-\beta}\Phi_\beta(t), && \beta > -1,
\end{aligned}$$

and define $\Psi_\alpha(t)$, $\psi_\alpha(t)$, $\Theta_\alpha(t)$ and $\theta_\alpha(t)$ in a similar way.

1.2. In 1930 it was proved by Bosanquet [**3**] that if $\phi_\alpha(t) \to 0$, as $t \to 0$, for $\alpha > 0$, then the Fourier series of $f(\theta)$, at $\theta = x$, is summable $(C, \alpha+\delta)$ for every $\delta > 0$. This result however breaks down for $\delta = 0$ [**15**]. For the case $\alpha = 0$, Hardy and Littlewood [**8**] obtained the convergence of the Fourier series (1.1.1), at $\theta = x$, under a set of stricter conditions, namely

$$\text{(i)} \qquad \phi(t) = o\left(\frac{1}{\log 1/t}\right), \quad \text{as} \quad t \to 0,$$

and

$$\text{(ii)} \qquad A_n = O(n^{-\delta}), \qquad \text{for some } \delta > 0.$$

They have also shown that their result breaks down if condition (ii)

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is omitted. They however replaced condition (i) by a weaker condition

$$\text{(i)}' \qquad \int_0^t |\phi(u)|\, du = o\left(\frac{t}{\log 1/t}\right), \quad \text{as} \quad t \to 0 \ [9].$$

In 1947 Wang [16] obtained the summability (C, α) of the Fourier series for every $\alpha > 0$, whenever

$$\text{(iii)} \qquad \phi_\alpha(t) = o\left(\frac{1}{\log 1/t}\right), \quad \text{as} \quad t \to 0,$$

the order condition on the Fourier coefficients being dropped. It is easy to see that the above condition can also be replaced by the weaker condition

$$\text{(iii)}' \qquad \int_0^t |\phi_\alpha(u)|\, du = o\left(\frac{t}{\log 1/t}\right), \quad \text{as} \quad t \to 0.$$

Recently Mohanty and Nanda [12] have proved that if $\theta(t) = o(1/\log 1/t)$, as $t \to 0$, and a_n, $b_n = O(n^{-\delta})$, $\delta > 0$, then $nB_n(x) \to (2l/\pi)(C, 1)$. Further, putting $l = 0$ and supposing that the conjugate function

$$\frac{1}{\pi} \int_{\to 0}^{\pi} \psi(t) \cot \frac{t}{2}\, dt$$

exists as a Cauchy integral at the origin, they obtained the convergence of the conjugate series, a result due to Hardy and Littlewood [8].

Analogous results for the $(C, \alpha+1)$ summability of the sequence $\{nB_n(x)\}$ and the (C, α) summability of the conjugate series, for $\alpha > 0$, have been obtained recently by the authoress [13]. The object of the present paper is to obtain similar results for the case $-1 < \alpha < 0$.

Just as with condition (i) Hardy and Littlewood required a stronger order condition on the Fourier coefficients for the convergence of the Fourier series, similarly with the condition (iii)$'$ for $-1 < \alpha < 0$, we require a stronger *nonlocal* condition than is necessary,[1] for the (C, α) summability of the Fourier series, namely $A_n = O(n^{\alpha-\delta})$, for some $\delta > 0$, with a similar remark applying to the summability (C, α) of the conjugate series. It is our conjecture that if we drop δ in the

[1] It is proved by Bosanquet and Offord [6] and Bhatnagar [1] that $A_n = o(n^\alpha)$ or $B_n = o(n^\alpha)$ whenever the Fourier series or the conjugate series is summable (C, α), for $-1 < \alpha < 0$, and these are the only nonlocal conditions needed for such summabilities.

above condition we will fail to obtain summability (C, α) of the Fourier series under condition (iii)$'$.

We devote §2 for obtaining criteria for summability $(C, \alpha+1)$ of the sequence $\{nB_n(x)\}$ and for summability (C, α) of the conjugate series, for $-1<\alpha<0$. In §3 we obtain similar results for the summability of the sequence $\{nA_n(x)\}$ and of the Fourier series.

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2.1. THEOREM 1. *If, for* $-1<\alpha<0$,

$$\Theta_\alpha^*(t) = \int_0^t |\, d\Theta_{\alpha+1}(u)\,| = o\left(\frac{t^{\alpha+1}}{\log 1/t}\right) \qquad \text{as } t \to 0,$$

and

$$B_n = O(n^{\alpha-\delta}), \qquad \delta > 0,$$

then the sequence $\{nB_n(x)\}$ *is summable* $(C, \alpha+1)$ *to sum* $2l/\pi$.

The following lemmas will be required for the proof of Theorem 1:

LEMMA 1 [1]. *If, for* $-1<\alpha<1$, $\bar{k}^\alpha(n, t)$ *denotes the* (C, α) *mean of the series*

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \sin nt$$

and if

$$\overline{K}^\alpha(n, t) = -\frac{2}{\pi}\,\frac{1}{A_n^\alpha}\,\frac{\cos(n; t)}{(2\sin t/2)^{\alpha+1}},$$

where $A_n^\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$ *and* $(n;t) = (n+1/2+\alpha/2)t - \alpha\pi/2$, *then for* $0<t<\pi$

$$\left|\bar{k}^\alpha(n, t) - \overline{K}^\alpha(n, t) - \frac{1}{\pi}\cot\frac{t}{2}\right| < An^{-1}t^{-2}.$$

LEMMA 2. *If* $k^\alpha(n, t)$ *and* $\bar{k}^\alpha(n, t)$ *denote the* (C, α) *means of the series*

$$\frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \cos nt$$

and

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \sin nt,$$

respectively, and if Δu_n *denotes* $u_n - u_{n-1}$, *then*

$$(2.1.1)\qquad \begin{aligned} n\Delta\bar{k}^{\alpha+1}(n,t) &= -\frac{\alpha+1}{n+\alpha+1}\frac{d}{dt}k^{\alpha}(n,t) \\ &= (\alpha+1)[\bar{k}^{\alpha}(n,t) - \bar{k}^{\alpha+1}(n,t)]; \ [2] \end{aligned}$$

$$(2.1.2)\qquad k^{\alpha}(n,t),\ \bar{k}^{\alpha}(n,t),\ n\Delta\bar{k}^{\alpha+1}(n,t) = O(n), \quad \textit{for } 0 < t < \pi,\ \alpha > -1;\ [2;5]$$

$$\left.\begin{aligned} &(2.1.3)\quad \bar{k}^{\alpha}(n,t) \\ &(2.1.4)\quad n\Delta\bar{k}^{\alpha+1}(n,t) \end{aligned}\right\} = O(n^{-\alpha}t^{-\alpha-1}), \qquad \textit{for } nt > 1,\ -1 < \alpha < 0.$$

PROOF OF (2.1.3), (2.1.4). From Lemma 1 we have

$$\begin{aligned} \bar{k}^{\alpha}(n,t) &= O[\overline{K}^{\alpha}(n,t)] + O(1/t) + O(n^{-1}t^{-2}) \\ &= O(n^{-\alpha}t^{-\alpha-1}) + O(1/t), \quad \text{if } nt > 1 \\ &= O(n^{-\alpha}t^{-\alpha-1}), \quad \text{if } -1 < \alpha < 0 \quad \text{and} \quad nt > 1. \end{aligned}$$

Also from (2.1.1) and (2.1.3) we have

$$\begin{aligned} n\Delta\bar{k}^{\alpha+1}(n,t) &= O\{\bar{k}^{\alpha}(n,t)\} \\ &= O(n^{-\alpha}t^{-\alpha-1}), \end{aligned}$$

which gives the required result.

LEMMA 3. *If, for* $-1<\alpha<0$,

$$L(n,u) = \frac{1}{\Gamma(-\alpha)}\int_u^X (t-u)^{-\alpha-1}n\Delta\bar{k}^{\alpha+1}(n,t)dt,$$

then

$$|L(n,u)| \begin{cases} < An^{\alpha+1}, & u \leqq 1/n, \\ < Au^{-\alpha-1}, & u \geqq 1/n, \end{cases}$$

the results holding uniformly in X.

PROOF OF LEMMA 3. We have, if $u+(1/n)<X$,[2]

$$\begin{aligned} L(n,u) &= \frac{1}{\Gamma(-\alpha)}\int_u^X (t-u)^{-\alpha-1}n\Delta\bar{k}^{\alpha+1}(n,t)dt \\ &= \frac{1}{\Gamma(-\alpha)}\left\{\int_u^{u+1/n} + \int_{u+1/n}^X\right\} \\ &= L_1 + L_2, \end{aligned}$$

[2] If $u+1/n \geqq X$, the integral need not be split up.

say. If $nu>1$, then from (2.1.4) we have

$$L_1 = O\left[\int_u^{u+(1/n)} (t-u)^{-\alpha-1} n^{-\alpha} t^{-\alpha-1} dt\right] = O[n^{-\alpha}(1/n)^{-\alpha}u^{-\alpha-1}]$$
$$= O[u^{-\alpha-1}].$$

Also by (2.1.1), (2.1.3) and by the second mean value theorem,

$$L_2 = -\frac{\alpha+1}{n+\alpha+1}\frac{1}{\Gamma(-\alpha)}\int_{u+(1/n)}^{X} (t-u)^{-\alpha-1}\frac{d}{dt}k^{\alpha}(n,t)dt$$
$$= O\left[n^{\alpha}\int_{u+(1/n)}^{\zeta}\frac{d}{dt}k^{\alpha}(n,t)dt\right] = O[n^{\alpha}n^{-\alpha}t^{-\alpha-1}]_{t=u+(1/n)}$$
$$= O[u^{-\alpha-1}].$$

Thus for $u>1/n$,

$$|L(n,u)| = O[u^{-\alpha-1}].$$

Next for the case $nu \leqq 1$, we have from (2.1.2)

$$L_1 = O\left[n\int_u^{u+(1/n)} (t-u)^{-\alpha-1}dt\right] = O[n^{\alpha+1}].$$

Again from (2.1.1), proceeding as before, we obtain

$$L_2 = O[n^{\alpha}\{k^{\alpha}(n,t)\}_{t=u+(1/n)}^{\zeta}]$$
$$= O[n^{\alpha+1}],$$

which gives finally

$$L(n,u) = O[n^{\alpha+1}].$$

LEMMA 4 [4]. *If* $-1 \leqq \alpha < 0$, $\beta > \alpha$ *and* $\phi(t)$ *is a function such that* (i) $\Phi_{\alpha+1}(t)$ *is a function of bounded variation in an interval* $(0, \eta)$ *and* (ii) $\Phi_{\alpha+1}(+0)=0$, *then* $\Phi_{\beta+1}(t)$ *is a Lebesgue integral,* $\Phi_{\beta+1}(+0)=0$, *and, for almost all* t *in* $(0, \eta)$,

$$\Phi_{\beta}(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}d\Phi_{\alpha+1}(u).$$

LEMMA 5 [10; 11]. *If* $\tau_{\alpha}(n)$ *and* $C_{\alpha}(n)$ *denote the* (C, α) *means of the sequence* $\{nU_n\}$ *and the series* $\sum U_n$, *respectively, then for* $\alpha > -1$,

$$\tau_{\alpha}(n) = n\{C_{\alpha}(n) - C_{\alpha}(n-1)\} = nC'_{\alpha}(n),$$
$$\tau_{\alpha+1}(n) = (\alpha+1)\{C_{\alpha}(n) - C_{\alpha+1}(n)\}.$$

2.2. Proof of Theorem 1. We have, since

$$B_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt dt,$$

$$\tau_{\alpha+1}(n) = n\Delta\overline{C}_{\alpha+1}(n) = \int_0^\pi \psi(t) n\Delta\bar{k}^{\alpha+1}(n, t)dt,$$

where $\tau_{\alpha+1}(n)$ and $\overline{C}_{\alpha+1}(n)$ denote the $(C, \alpha+1)$ mean of the sequence $\{nB_n(x)\}$ and the series $\sum B_n(x)$, respectively. Now using the fact that if $\psi(t)=1$ in $(0, \pi)$ then the sequence $\{nB_n(x)\}$ tends to the limit $2/\pi$, in Cesàro mean of order δ, for every $\delta>0$, we have

$$\tau_{\alpha+1}(n) - \frac{2l}{\pi} = \int_0^\pi \theta(t) n\Delta\bar{k}^{\alpha+1}(n, t)dt$$

$$= \left\{\int_0^{n^{-r}} + \int_{n^{-r}}^\pi\right\}\theta(t) n\Delta\bar{k}^{\alpha+1}(n, t)dt, \qquad n < 1, \tag{2.2.1}$$

$$= J + I,$$

say. Now by Lemma 4,

$$J = \frac{1}{\Gamma(-\alpha)}\int_0^{n^{-r}} n\Delta\bar{k}^{\alpha+1}(n, t)\left\{\int_0^t (t-u)^{-\alpha-1}d\Theta_{\alpha+1}(u)\right\} dt$$

$$= \frac{1}{\Gamma(-\alpha)}\int_0^{n^{-r}} d\Theta_{\alpha+1}(u)\left\{\int_u^{n^{-r}} (t-u)^{-\alpha-1} n\Delta\bar{k}^{\alpha+1}(n, t)dt\right\}$$

$$= \int_0^{n^{-r}} d\Theta_{\alpha+1}(u) L(n, u),$$

where

$$L(n, u) = \frac{1}{\Gamma(-\alpha)}\int_u^{n^{-r}} (t-u)^{-\alpha-1} n\Delta\bar{k}^{\alpha+1}(n, t)dt.$$

Or

$$J = \left\{\int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}}\right\} d\Theta_{\alpha+1}(u) L(n, u) = J_1 + J_2, \tag{2.2.2}$$

say. It follows from Lemma 3 and the first condition of Theorem 1, that

$$J_1 = O\left[\int_0^{n^{-1}} n^{\alpha+1}\left| d\Theta_{\alpha+1}(u)\right|\right]$$

$$= o\left[n^{\alpha+1}\frac{u^{\alpha+1}}{\log 1/u}\right]_{u=1/n} = o(1), \tag{2.2.3}$$

and

$$
\begin{aligned}
J_2 &= O\left[\int_{n^{-1}}^{n^{-r}} u^{-\alpha-1} \left| d\Theta_{\alpha+1}(u) \right| \right] \\
&= O[\{\Theta_\alpha^*(u) u^{-\alpha-1}\}_{u=1/n}^{n^{-r}}] \\
&\quad + O\left[\int_{n^{-1}}^{n^{-r}} \Theta_\alpha^*(u) u^{-\alpha-2} du\right] \qquad (2.2.4) \\
&= o(1) + o\left[\int_{n^{-1}}^{n^{-r}} \frac{1}{u \log 1/u}\, du\right] \\
&= o(1) + o[\{\log \log u\}_{u=1/n}^{n^{-r}}] \\
&= o(1).
\end{aligned}
$$

Next by (2.1.1),

$$
\begin{aligned}
I &= \int_{n^{-r}}^{\pi} \theta(t) n \Delta \bar{k}^{\alpha+1}(n, t) dt \\
&= (\alpha+1) \int_{n^{-r}}^{\pi} \theta(t) \{\bar{k}^\alpha(n, t) - \bar{k}^{\alpha+1}(n, t)\} dt \\
&= (\alpha+1) \int_{n^{-r}}^{\pi} \theta(t) \left\{\bar{k}^\alpha(n, t) - \overline{K}^\alpha(n, t) - \frac{1}{\pi} \cot \frac{t}{2}\right\} dt \\
&\quad - (\alpha+1) \int_{n^{-r}}^{\pi} \theta(t) \left\{\bar{k}^{\alpha+1}(n, t) - \overline{K}^{\alpha+1}(n, t) - \frac{1}{\pi} \cot \frac{t}{2}\right\} dt \qquad (2.2.5) \\
&\quad + (\alpha+1) \int_{n^{-r}}^{\pi} \theta(t) \{\overline{K}^\alpha(n, t) - \overline{K}^{\alpha+1}(n, t)\} dt \\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

say, where

$$
\overline{K}^\alpha(n, t) = -\frac{2}{\pi} \frac{1}{A_n^\alpha} \frac{\cos(n; t)}{(2 \sin t/2)^{\alpha+1}},
$$

$$
A_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}
$$

and $(n; t)$ is written for $(n+\frac{1}{2}+\alpha/2)-\alpha\pi/2$.

Now from Lemma 1, writing $\theta^*(t) = \int_0^t |\theta(u)| du = o(t/(\log 1/t))$, we have

$$
\begin{aligned}
I_1 &= O\left[\int_{n^{-r}}^{\pi} |\theta(t)|\, n^{-1}t^{-2}dt\right] \\
&= O[\{n^{-1}t^{-2}\theta^*(t)\}_{t=n^{-r}}^{\pi}] \\
&\quad + O\left[n^{-1}\int_{n^{-r}}^{\pi} \theta^*(t)t^{-3}dt\right] \qquad (2.2.6)\\
&= o(1) + o\left[n^{-1}\int_{n^{-r}}^{\pi} \frac{1}{t^2 \log 1/t}\, dt\right] \\
&= o(n^{r-1}/\log n) = o(1).
\end{aligned}
$$

It can be shown similarly that

$$I_2 = o(1). \qquad (2.2.7)$$

Next we have

$$
\begin{aligned}
I_3 &= (\alpha+1)\int_{n^{-r}}^{\pi} \psi(t)\overline{K}^{\alpha}(n,t)dt \\
&\quad - (\alpha+1)\int_{n^{-r}}^{\pi} \psi(t)\overline{K}^{\alpha+1}(n,t)dt \\
&\quad - (\alpha+1)l\int_{n^{-r}}^{\pi} \overline{K}^{\alpha}(n,t)dt \qquad (2.2.8)\\
&\quad + (\alpha+1)l\int_{n^{-r}}^{\pi} \overline{K}^{\alpha+1}(n,t)dt \\
&= I_{3,1} - I_{3,2} - I_{3,3} + I_{3,4},
\end{aligned}
$$

say. Now by the second mean value theorem,

$$
\begin{aligned}
I_{3,3} &= \frac{2l}{\pi}(\alpha+1)\int_{n^{-r}}^{\pi} \frac{1}{A_n^{\alpha}} \frac{\cos\{(n+(\alpha+1)/2)t - \alpha\pi/2\}}{(2\sin t/2)^{\alpha+1}}\, dt \\
&= O[n^{-\alpha}n^{r(\alpha+1)}n^{-1}] \qquad (2.2.9)\\
&= o(1).
\end{aligned}
$$

Similarly,

$$I_{3,4} = o(1). \qquad (2.2.10)$$

Again

$$
\begin{aligned}
I_{3,1} &= (\alpha+1)\sum_{\nu=1}^{\infty}\int_{n^{-r}}^{\pi} B_\nu(x)\sin \nu t\overline{K}^{\alpha}(n,t)\,dt \\
&= -\frac{2}{\pi}\,\frac{\alpha+1}{A_n^{\alpha}}\sum_{\nu=1}^{\infty}\Bigg[B_\nu(x)\int_{n^{-r}}^{\pi}\sin \nu t \\
&\qquad\qquad \cdot\frac{\cos\{(n+(\alpha+1)/2)t-\alpha\pi/2\}}{(2\sin t/2)^{\alpha+1}}\,dt\Bigg] \\
&= -\frac{1}{\pi}\,\frac{\alpha+1}{A_n^{\alpha}}\sum_{\nu=1}^{\infty}\Bigg\{B_\nu(x)\left(2\sin\frac{1}{2n^r}\right)^{-\alpha-1} \\
&\qquad\qquad \cdot\int_{n^{-r}}^{\delta}\Bigg[\sin\left\{\left(\nu+n+\frac{\alpha+1}{2}\right)t-\frac{\alpha\pi}{2}\right\} \\
&\qquad\qquad -\sin\left\{\left(n-\nu+\frac{\alpha+1}{2}\right)t-\frac{\alpha\pi}{2}\right\}\Bigg]dt\Bigg\},\ (n^{-r}\leqq\delta<\pi) \qquad (2.2.11)\\
&= O\left[n^{r(\alpha+1)-\alpha}\sum_{\nu=1}^{\infty}\frac{B_\nu(x)}{n+\nu+(\alpha+1)/2}\right] \\
&\quad + O\left[n^{r(\alpha+1)-\alpha}\sum_{\nu=1}^{\infty}\frac{B_\nu(x)}{n-\nu+(\alpha+1)/2}\right] \\
&= O\left[n^{r(\alpha+1)-\alpha}\sum_{\nu=1}^{\infty}\frac{\nu^{\alpha-\delta}}{|\nu-(n+(\alpha+1)/2)|}\right] \\
&= O\left[n^{r(\alpha+1)-\alpha}\sum_{\nu=1}^{n}\frac{\nu^{\alpha-\delta}}{n+(\alpha+1)/2-\nu}\right] \\
&\quad + O\left[n^{r(\alpha+1)-\alpha}\sum_{\nu=n+1}^{\infty}\frac{\nu^{\alpha-\delta}}{\nu-(n+(\alpha+1)/2)}\right] = P_1+P_2,
\end{aligned}
$$

say. Now

$$
\begin{aligned}
P_1 &= O\left[n^{r(\alpha+1)-\alpha}\sum_{1\leqq\nu\leqq n/2}\left\{\frac{\nu^{\alpha-\delta}}{n+(\alpha+1)/2-\nu}\right\}\right] \\
&\quad + O\left[n^{r(\alpha+1)-\alpha}\sum_{n/2\leqq\nu\leqq n}\left\{\frac{\nu^{\alpha-\delta}}{n+(\alpha+1)/2-\nu}\right\}\right] \\
&= O\left[\frac{n^{r(\alpha+1)-\alpha}}{(n+\alpha+1)/2}\sum_{1\leqq\nu\leqq n/2}\nu^{\alpha-\delta}\right] \qquad (2.2.12)\\
&\quad + O\left[n^{r(\alpha+1)-\alpha-\delta+\alpha}\sum_{n/2\leqq\nu\leqq n}\left\{\frac{1}{n+(\alpha+1)/2-\nu}\right\}\right] \\
&= O\left[n^{r(\alpha+1)-\alpha-1+\alpha+1-\delta}\right]+O\left[n^{r(\alpha+1)-\delta}\log n\right] = o(1),
\end{aligned}
$$

choosing $r<\delta/(\alpha+1)$.

Again breaking P_2 in two parts, we have, proceeding as before,

$$
\begin{aligned}
P_2 &= O\left[n^{r(\alpha+1)-\alpha} \sum_{n+1\leq\nu\leq 2n} \left\{\frac{\nu^{\alpha-\delta}}{\nu-(n+(\alpha+1)/2)}\right\}\right] \\
&\quad + O\left[n^{r(\alpha+1)-\alpha} \sum_{\nu\geq 2n} \left\{\frac{\nu^{\alpha-\delta}}{\nu-(n+(\alpha+1)/2)}\right\}\right] \qquad (2.2.13)\\
&= O[n^{r(\alpha+1)-\delta}] + O[n^{r(\alpha+1)-\delta}\log n] \\
&= o(1).
\end{aligned}
$$

It follows from (2.2.11), $\cdots$, (2.2.13), that

$$I_{3,1} = o(1). \qquad (2.2.14)$$

With exactly similar arguments it can be shown that

$$I_{3,2} = o(1). \qquad (2.2.15)$$

Collecting all the results (2.2.1), $\cdots$, (2.2.15), we obtain

$$\tau_{\alpha+1}(n) = \frac{2l}{\pi} + o(1).$$

This completes the proof of Theorem 1.

2.3. Lemma 6 [9]. *If $\sum u_n$ is summable (A), then a necessary and sufficient condition that it should be summable (C, k), for $k>-1$, is that the sequence $\{nu_n\}$ is summable $(C, k+1)$ to the value zero.*

As direct application of Theorem 1 and Lemma 6 and the fact that the existence of the conjugate function

$$\frac{1}{\pi}\int_{\to 0}^{\pi} \psi(t)\cot\frac{t}{2}\,dt$$

is a necessary condition for the negative order summability of the conjugate series [1] and that it implies summability A of the conjugate series, we obtain the following theorem:

Theorem 2. *If, for $-1<\alpha<0$,*

$$\int_0^t |d\Psi_{\alpha+1}(u)| = o\left(\frac{t^{\alpha+1}}{\log 1/t}\right), \quad as \quad t\to 0,$$

and

$$B_n = O(n^{\alpha-\delta}), \qquad \delta>0,$$

then the necessary and sufficient condition that the conjugate series (1.1.2), *at* $\theta=x$, *may be summable* (C, α), $-1<\alpha<0$, *to the value*

$$\frac{1}{\pi}\int_{\to 0}^{\pi}\psi(t)\cot\frac{t}{2}\,dt,$$

is that the above integral exists as a Cauchy integral at the origin.

2.4. With the help of Lemma 5, the following theorem can be established.[3]

THEOREM 3. *If, for* $-1<\alpha<0$,

$$\int_0^t |\,d\Theta_{\alpha+1}(u)\,| = o\left(\frac{t^{\alpha+1}}{\log 1/t}\right), \quad as \quad t\to 0,$$

and if

$$B_n = O(n^{\alpha-\delta}), \qquad \delta > 0,$$

then

$$\lim_{n\to\infty}\,[\overline{C}_\alpha(\lambda n) - \overline{C}_\alpha(n)] = \frac{2l}{\pi}\log\lambda, \qquad \lambda > 1,$$

where $\overline{C}_\alpha(n)$ *denotes the Cesàro mean of order* α *of the conjugate series* (1.1.2), *at* $\theta=x$.

3.1. We shall now prove the analogous theorem for the (C, α) summability of the Fourier series for $-1<\alpha<0$.

THEOREM 4. *If, for* $-1<\alpha<0$,

$$\int_0^t |\,d\Phi_{\alpha+1}(u)\,| = o\left(\frac{t^{\alpha+1}}{\log 1/t}\right), \tag{3.1.1}$$

as $t\to 0$, *and if*

$$A_n = O(n^{\alpha-\delta}), \qquad \delta > 0,$$

then the Fourier series (1.1.1), *at* $\theta=x$, *is summable* (C, α) *to sum* s.

The following lemmas will be required for the proof of Theorem 4.

LEMMA 7 [7]. *If* $k^\alpha(n, t)$ *denotes the* (C, α) *mean of the series*

$$\frac{1}{\pi}+\frac{2}{\pi}\sum_{\nu=1}^{\infty}\cos\nu t,$$

[3] See [13, Theorem 4].

and if

$$K^{\alpha}(n,t) = \frac{2}{\pi}\,\frac{1}{A_n^{\alpha}}\,\frac{\sin(n;t)}{(2\sin t/2)^{\alpha+1}},$$

where

$$A_n^{\alpha} = \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}, \qquad (n;t) = \left(n+\frac{1}{2}+\frac{\alpha}{2}\right)t - \frac{\alpha\pi}{2},$$

then for $-1<\alpha<1$,

$$\left| k^{\alpha}(n,t) - K^{\alpha}(n,t) \right| < An^{-1}t^{-2}.$$

LEMMA 8. *If, for* $-1<\alpha<0$,

$$M(n,u) = \frac{1}{\Gamma(-\alpha)}\int_u^X (t-u)^{-\alpha-1} n\Delta k^{\alpha+1}(n,t)\,dt, \qquad r<1,$$

then

$$\left| M(n,u) \right| \begin{cases} < An^{\alpha+1}, & u \leqq 1/n, \\ < Au^{-\alpha-1}, & u \geqq 1/n, \end{cases}$$

the results holding uniformly in X.

The proof of Lemma 8 is exactly similar to that of Lemma 3.

3.2. PROOF OF THEOREM 4. Writing $\bar{\Phi}(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$ and denoting by $\bar{\tau}_{\alpha+1}(n)$ the $(C, \alpha+1)$ mean of the sequence $\{nA_n(x)\}$, we have proceeding as in the proof of Theorem 1,

$$\bar{\tau}_{\alpha+1}(n) = \int_0^{\pi} \bar{\phi}(t) n\Delta k^{\alpha+1}(n,t)\,dt = \int_0^{\pi} \phi(t) n\Delta k^{\alpha+1}(n,t)\,dt$$
$$= \int_0^{n^{-r}} + \int_{n^{-r}}^{\pi} = J + I,$$

say. Now it is easily seen that

$$J = \int_0^{n^{-r}} d\Phi_{\alpha+1}(u) M(n,u),$$

where

$$M(n,u) = \frac{1}{\Gamma(-\alpha)}\int_u^{n^{-r}} (t-u)^{-\alpha-1} n\Delta k^{\alpha+1}(n,t)\,dt.$$

Now taking help from Lemma 8, we obtain $J=o(1)$ as in the proof of Theorem 1. Also

$$I = \int_{n^{-r}}^{\pi} \phi(t) n \Delta k^{\alpha+1}(n, t) dt = (\alpha+1) \int_{n^{-r}}^{\pi} \phi(t) \{ k^{\alpha}(n, t) - k^{\alpha+1}(n, t) \} dt$$

$$= (\alpha+1) \int_{n^{-r}}^{\pi} \phi(t) \{ k^{\alpha}(n, t) - K^{\alpha}(n, t) dt \}$$

$$- (\alpha+1) \int_{n^{-r}}^{\pi} \phi(t) \{ k^{\alpha+1}(n, t) - K^{\alpha+1}(n, t) \} dt$$

$$+ (\alpha+1) \int_{n^{-r}}^{\pi} \phi(t) \{ K^{\alpha}(n, t) - K^{\alpha+1}(n, t) \} dt = o(1),$$

as before. This shows that under the condition of the theorem the sequence $\{nB_n(x)\} \to O(C, \alpha+1)$. The proof now follows from the fact that the hypotheses imply the summability (C) of (1.1.1), at $\theta = x$, and therefore also summability (A) and hence by Lemma 7 summability (C, α).

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