MATRIX \( \mathbb{N} \)-RINGS

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1. If \( R \) is a regular ring with unit element, the lattice \( L_R(\mathcal{R}_R) \) of principal left (right) ideals ordered by inclusion is a complemented modular lattice and the lattices \( L_R \) and \( \mathcal{R}_R \) are dual isomorphic, [5, Part II, Chapter II]. \( R \) is called an \( \mathbb{N} \)-ring if \( L_R \) is \( \mathbb{N} \)-complete and \( \mathbb{N} \)-continuous and when \( R \) is an \( \mathbb{N} \)-ring for any cardinal number \( \mathbb{N} \), \( R \) is a von Neumann ring.

The ring \( S_n \) of \( n \times n \) matrices with entries in \( R \) is regular if \( R \) is regular, but the fact that \( R \) is an \( \mathbb{N} \)-ring does not guarantee that \( S_n \) is also an \( \mathbb{N} \)-ring. If \( S_n \) is an \( \mathbb{N} \)-ring (von Neumann ring) for every positive integer \( n \), we say that \( R \) is a matrix \( \mathbb{N} \)-ring (von Neumann ring). In the present note we give a necessary and sufficient condition for an \( \mathbb{N} \)-ring to be a matrix \( \mathbb{N} \)-ring, and two examples of matrix \( \mathbb{N} \)-rings.

As a consequence of the additivity of upper \( \mathbb{N} \)-continuity in \( \mathbb{N} \)-complete, complemented modular lattices (see [1, Theorem 4.3]) and the additivity of \( \mathbb{N} \)-completeness under certain conditions it was shown in [2] that \( R \) is a matrix \( \mathbb{N} \)-ring if \( R^2 \) is an \( \mathbb{N} \)-ring [2, Corollary 3 of Theorem 3.1].

2. In what follows \( R \) denotes a regular ring with unit element, \( (u)_l \) and \( (u)_r \) are the principal left and right ideal, respectively, generated by \( u \in R \). \( \Omega \) denotes an ordinal number and \( \Omega \) its cardinal power.

It is convenient to think of \( L_{R_\Omega} \) as the lattice of finitely generated submodules of the left \( R \)-module of ordered pairs \( (a_1, a_2) \), \( a_1 \in R \), [5, Part II, Chapter II, Appendix 3]. \( \{(a_1, a_2)\} \) will denote the submodule generated by \( (a_1, a_2) \). A finitely generated submodule \( M \) of the left \( R \)-module of ordered pairs admits a canonical basis, that is,

\[
M = \{(e, 0)\} \oplus \{(a, f)\},
\]

where \( e^2 = e, f^2 = f, fa = a, ae = 0 \) and \( \oplus \) means direct sum. The submodule \( \{(e, 0)\} \) is uniquely defined by \( M \) since it is equal to \( M \cap \{(1, 0)\} \), that is, \( \{(e, 0)\} \) is the submodule of elements of \( M \) whose second component is zero.

Our first step is to find a decomposition of \( M \) where the submodule \( M \cap \{(0, 1)\} \) also appears explicitly.

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**Lemma 1.** Any finitely generated submodule $M$ of the left $R$-module of ordered pairs $(a_1, a_2)$, $a_i \in R$, can be decomposed in the following way

$$M = \{(e_1, 0)\} \oplus \{(a_1, a_2)\} \oplus \{(0, e_2)\}$$

where $e_i = e_i$, $a_1e_1 = a_2e_2 = 0$ and $(a_1)_R = (a_2)_R$.

**Proof.** Let $U$ be the involutorial automorphism of the left $R$-module of ordered pairs which takes the vector $(a, b)$ into $(b, a)$. $U$ takes a finitely generated submodule into a finitely generated submodule and defines an involutorial automorphism $U$ of $L_R$.

Suppose $M = \{(e_1, 0)\} \oplus \{(a_i)\}$ under a canonical decomposition. Then $e_1a = 0$, $f_2 = a$. Taking a canonical decomposition of

$$U\{(a_i)\} = \{(f, a)\} = \{(e_2, 0)\} \oplus \{(a_2, a_1)\},$$

$e_2 = e_2$, $a_2e_2 = 0$ and $(a_1)_R = (a_2)_R$. Therefore

$$M = \{(e_1, 0)\} \oplus \{(a_1, a_2)\} \oplus \{(0, e_2)\}.$$ 

Since $M \cap \{(0, 1)\} = \{(a_i)\} \cap \{(0, 1)\}$, it follows that $M \cap \{(0, 1)\} = \{(0, e_2)\}$. Because $a_1 = xa$ and $ae_1 = 0$, we have $a_1e_1 = 0$. Moreover, $\{(a_1, a_2)\} \cap \{(0, 1)\} = \{(a_1, a_2)\} \cap \{(1, 0)\} = 0$ implies that the left annihilators of $a_1$ and $a_2$ coincide, hence $(a_1)_R = (a_2)_R$.

3. The next point which we need to emphasize is the fact that a left factor-correspondence ([5, Part II, Definition 15.1], notice that this definition of f.-c. is for right principal ideals which we call right factor-correspondence) between $(u)_1$ and $(v)_1$ can be determined by a pair of elements of $R$. This fact is an immediate consequence of the definition of factor-correspondence and its precise statement is given in the following lemma.

**Lemma 2.** A left factor-correspondence between $(u)_1$ and $(v)_1$ is completely determined by any pair of elements $u'$, $v' \in R$ corresponding to each other and such that $(u')_1 = (u)_1$ and $(v')_1 = (v)_1$. Conversely, if $(u')_1 = (v')_1$, the one-to-one mapping

$$xu' \leftrightarrow xv', \quad x \in R,$$

defines a factor-correspondence between $(u')_1$ and $(v')_1$.

The factor-correspondence defined by the pair $u, v$ will be denoted by $(u:v)$.

We introduce an order in the set of factor-correspondences by defining

$$(u_1:v_1) \geq (u_2:v_2)$$

if $(u_1)_1 \supset (u_2)_1$ and $u_2$ and $v_2$ correspond to each other in $(u_1:v_1)$. 

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Theorem 1. Let \( R \) be a regular ring such that \( L(R) \) is upper \( S \)-complete and upper \( S \)-continuous. Then the lattice \( L(R) \) is upper \( S \)-complete if and only if every increasing chain \( (u^\alpha : v^\alpha), \alpha < \Omega \) and \( \Omega \subseteq S \), of left factor-correspondences has a supremum. Moreover, if \( L(R) \) is \( S \)-complete it is upper \( S \)-continuous.

Proof. The last statement is an immediate consequence of the theorem of Amemiya and Halperin on the additivity of upper \( S \)-continuity is complemented, \( S \)-complete modular lattices, (cf. \[1, Theorem 4.3\]). For, \( L(R_2) = [0, \{(1, 0)\} \cup \{(0, 1)\}] \) and \( [0, \{(1, 0)\}] \) is isomorphic to \( L(R_1) \).

Assume that \( L(R_2) \) is upper \( S \)-complete and let \( (u^\alpha : v^\alpha), \alpha < \Omega \) and \( \Omega \subseteq S \), be an increasing chain of left factor-correspondences. Then the modules \( M^\alpha = \{(u^\alpha, v^\alpha)\} \) form an increasing \( \Omega \)-chain. Since \( M^\alpha \cap \{(1, 0)\} = M^\alpha \cap \{(0, 1)\} = 0 \) and \( (U(M^\alpha) \alpha < \Omega) \cap \{(0, 1)\} = 0 \), because \( L(R_2) \) is upper \( S \)-continuous. Therefore, by Lemma 1, \( U(M^\alpha) \alpha < \Omega = \{(u, v)\} \) with \( (u)_r = (v)_r \), and it is clear that \( (u:v) \) is the supremum of the \( (u^\alpha : v^\alpha) \).

So we assume now that, if \( \Omega \subseteq S \), every increasing \( \Omega \)-chain of left factor-correspondences has a supremum and proceed to show that \( L(R_2) \) is upper \( S \)-complete. Let \( M^\alpha \) be an increasing \( \Omega \)-chain of modules in \( L(R_2) \) and let

\[
M^\alpha = \{(e^\alpha, 0)\} \oplus \{(a^\alpha, f^\alpha)\}
\]

be a canonical decomposition of the \( M^\alpha \). If \( (e_1)_r = (U(e^\alpha) \alpha < \Omega) \), it is clear that the chain \( M^\alpha \) has a supremum if and only if the increasing chain of \( N^\alpha = M^\alpha \cup \{(e_1, 0)\} \) has a supremum. Now,

\[
N^\alpha = \{(e_1, 0)\} \oplus \{(b^\alpha, f^\alpha)\}
\]

with \( b^\alpha = a^\alpha - a^\alpha e_1 \), is a canonical decomposition of \( N^\alpha \) and we only need show that the chain of modules \( \{(b^\alpha, f^\alpha)\} \) has a supremum. This chain is actually increasing, for \( \{(b^\alpha, f^\alpha)\} \subset N^\beta \) for \( \alpha \leq \beta \) implies that

\[
(b^\alpha, f^\alpha) = d_1(e_1, 0) + d_2(b^\alpha, f^\alpha) = (d_1e_1 + d_2b^\alpha, d_2f^\alpha),
\]

hence \( b^\alpha = d_2b^\beta \), because \( b^\gamma e_1 = 0 \) for all \( \gamma \subset \Omega \), and consequently \( (b^\alpha, f^\alpha) = d_2(b^\beta, f^\beta) \). By Lemma 1,

\[
(b^\alpha, f^\alpha) = (b^1_1, b^1_2) \oplus (0, e_2^\alpha),
\]

where \( b^2_2 e_2^\alpha = 0 \) and \( (b^2_2)_r = (b^2_2)_r \). Now \( \{(b^\alpha, f^\alpha)\} \cap \{(1, 0)\} = 0 \) implies \( (b^2_2) \cap (e_2^\alpha)_1 = 0 \) if \( \alpha < \beta \); for it follows from \( d_2b_2^\alpha = d_2b^\beta \) that

\[
(d_1b_1^\alpha, 0) = d_1(b_1^\alpha, b_2^\alpha) - d_2(0, e_2^\alpha) \subset \{(b^\beta, f^\beta)\},
\]
hence \( ds b_s = 0 \) and consequently \( d_s b_s = 0 \), since \((b_s^2)_r = (b_s^2)_r\). If \((e_2)_i = \bigcup \{ (e_2^2)_i | \alpha < \Omega \} \), then, by the upper \( \mathbb{S} \)-continuity of \( L_\mathbb{R} \),

\[ (e_2)_i \cap (b_2)_i = 0. \]

The increasing chain defined by the modules \((1)\) has a supremum if and only if the increasing chain defined by

\[ P^* = \{(b_1, b_2^2)\} + \{(0, e_2)\} \]

has a supremum. Now \((2)\) implies \( P^* \cap \{(1, 0)\} = 0 \), hence, by Lemma 1,

\[ P^* = \{(a_1^\alpha, a_2^\alpha)\} \oplus \{(0, e_2)\}, \]

where \( a_1^\alpha e_2 = 0 \) and \( (a_1^\alpha)_r = (a_2^\alpha)_r \). It is easily seen that the modules \( \{(a_1^\alpha, a_2^\alpha)\}, \alpha < \Omega, \) form an increasing chain; therefore \( (a_1^\alpha: a_2^\alpha) \) is an increasing chain of left factor-correspondences. Let \( (a_1:a_2) \) be the supremum of this chain, then

\[ \{(a_1, a_2)\} \oplus \{(0, e_2)\} = \bigcup (P^* | \alpha < \Omega) = \bigcup \{ (b_i, f_i) \} | \alpha \in \Omega). \]

**Theorem 2.** Let \( \mathfrak{R} \) be an \( \mathbb{S} \)-ring. Then \( \mathfrak{R}_9 \) is an \( \mathbb{S} \)-ring if and only if every increasing \( \Omega \)-chain of left or right factor-correspondences has a supremum when \( \mathfrak{R} \leq \mathbb{S} \).

**Proof.** This is a consequence of Theorem 1 and the dual isomorphism between \( L_\mathfrak{R}_9 \) and \( R_\mathfrak{R}_9 \).

**Corollary 1.** Any complete rank-ring \( \mathfrak{R} \) is a matrix von Neumann ring. (See [5, Part II, Definition 18.1].)

**Proof.** Given any increasing chain of left factor-correspondences \( (u^\alpha:v^\alpha), \alpha < \Omega, \) the \( u^\alpha \) can be chosen to be idempotents. Let \( R \) denote the rank-function. Since \( R(u^\alpha) \) is a bounded increasing chain of real numbers, we can replace the given chain by an increasing sequence \( (u^\alpha_i, v^\alpha_i), i = 1, 2, \ldots, \) with the same upper bound. Then we can even assume that the \( u^\alpha_i \) besides being idempotent satisfy \( u^\alpha_i u^\alpha_i = u^\alpha_i \) for \( i < j \) (we only have to apply the construction in the proof of Lemma 18.3 of [5] to the sequence \( 1 - u^\alpha_i \)). Now, since \( (u^\alpha_i - u^\alpha_i) u^\alpha_i = u^\alpha_i (u^\alpha_i - u^\alpha_i) = 0 \) and \( (u^\alpha_i - u^\alpha_i)^2 = u^\alpha_i - u^\alpha_i \),

\[ R(u^\alpha_i - u^\alpha_i) = R(u^\alpha_i) - R(u^\alpha_i). \]

Therefore, by the completeness of \( \mathfrak{R} \), the Cauchy sequence \( u^\alpha_i, i = 1, 2, \ldots \) has a limit \( u \). On the other hand, \( R(v^\alpha_i - v^\alpha_i) = R(u^\alpha_i - u^\alpha_i) \) since \( v^\alpha_i - v^\alpha_i \) and \( u^\alpha_i - u^\alpha_i \) correspond to each other under \( (u^\alpha_i:v^\alpha_i) \). Then, if \( \lim_{i \to \infty} v^\alpha_i = v, (u:v) \) is the supremum of the given chain.
Corollary 2. If $\mathfrak{R}$ is an $\mathfrak{N}$-ring and $\mathcal{L}_{\mathfrak{R}}$ has a large 2 basis, then every increasing chain of left or right factor-correspondences in $\mathfrak{R}$ has a supremum.

Proof. The corollary follows from Theorem 2 and [2, Corollary 2 of Theorem 3.1].

4. As an application of Theorem 2 we give two examples of matrix $\mathfrak{N}$-rings.

Example 1. (This is a generalization of Kaplansky's example [3, p. 526] and [4, Example 3, p. 604].) Let $J$ be any set such that $J \subseteq \mathbb{N}$. Let $\{D_\alpha\}_{\alpha \in J}$ be a family of division rings and $F_\alpha$ a proper division subring of $D_\alpha$ for every $\alpha \in J$. Consider the functions $f$ which map each element $\alpha \in J$ into an element of $D_\alpha$ and such that, if $J_f = \{\alpha \mid \alpha \in J, f(\alpha) \in F_\alpha\}$, $J_f \subseteq \mathbb{N}$. Then the ring $\mathfrak{R}$ of such functions under the natural definition of addition and multiplication is a von Neumann ring. Applying Theorem 2 it is easily seen that $\mathfrak{R}$ is a matrix $\mathfrak{N}$-ring, but, if $J > \mathbb{N}$, $\mathfrak{R}$ is not a matrix $\mathfrak{N}'$-ring for any $\mathfrak{N}' > \mathbb{N}$.

Example 2. Let $\mathfrak{B}$ be an $\mathfrak{N}$-complete Boolean algebra and $X$ its dual space, that is, $X$ is the space of the Stone representation. Then $X$ is a totally disconnected, compact, Hausdorff space. Consider the functions $f$ over $X$ with values in a Galois field $F$ satisfying the condition: for every $a \in F$, the set

$$X_a = \{x \mid x \in X, f(x) = a\}$$

is a clopen set of $X$. Then, under the natural definition of addition and multiplication of a function, such functions form a matrix $\mathfrak{N}$-ring.

References

3. I. Kaplansky, Any orthocomplemented complete modular lattice is a continuous geometry, Ann. of Math. (2) 61 (1955), 524–541.

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