ON THE ABSOLUTE HARMONIC SUMMABILITY OF THE FACTORED FOURIER SERIES

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1. Let $\{s_n\}$ denote the nth partial sum of the series $\sum a_n$. We write

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} \frac{1}{n-\nu+1} s_\nu, \tag{1.1}$$

where

$$P_n = \sum_{\nu=0}^{n} \frac{1}{\nu+1} \sim \log n. \tag{1.2}$$

The series $\sum a_n$ is said to be absolutely Harmonic summable if the sequence $\{t_n\}$ is of bounded variation, that is to say, the infinite series

$$\sum | t_n - t_{n-1} |$$

is convergent. L. McFadden [2] has proved that the method is absolutely regular and implies absolute Cesàro summability of every positive order.

2. Let $f(t)$ be a periodic function with period 2π and integrable L over $(-\pi, \pi)$. Without any loss of generality the constant term in the Fourier series can be taken to be zero, so that

$$\begin{aligned} f(t) &\sim \sum (a_n \cos nt + b_n \sin nt) \\ &\equiv \sum A_n(t); \end{aligned} \tag{2.1}$$

and

$$\int_{-\pi}^{\pi} f(t)dt = 0.$$

We write

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\};$$

and

$$\phi_1(t) = \frac{1}{t} \int_0^t \phi(u)du.$$

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Varshney [5] has considered the absolute Harmonic summability of the factored Fourier series

$$\sum A_n(t)/\log(n+1) \tag{2.2}$$

under Jordan's criterion of convergence for Fourier series. He has proved the following

THEOREM A. *If $\phi(t)$ is of bounded variation in $(0, \pi)$, then the series (2.2) is absolutely Harmonic summable.*

The object of the present paper is to study the absolute Harmonic summability of a series related to a Fourier series under de la Vallée-Poussin's criterion for convergence of the Fourier series which is known to be less stringent than Jordan's condition in view of the fact that the former includes the latter one. More precisely we establish the following:

THEOREM 1. *If $\phi_1(t)$ is of bounded variation in $(0, \pi)$, then the factored Fourier series*

$$\sum A_n(t) \log(n+1)\lambda_n/n,$$

where $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, is absolutely Harmonic summable at the point $t=x$.

In order to prove the above theorem we establish an absolute Harmonic summability factor theorem for general infinite series.

THEOREM 2. *If*

$$B_n \equiv \sum_{\nu=1}^{n} \nu a_\nu = O(n)$$

then the series $\sum a_n \log(n+1)\lambda_n/n$, where $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, is absolutely Harmonic summable.

Since a Lebesgue integral is absolutely continuous, it is plain that $\phi_1(t)$ is of bounded variation in any range (δ, π), $\delta>0$. Thus a very interesting consequence of Theorem 1 is that the *absolute Harmonic summability of the series $\sum A_n(t) \log(n+1)\lambda_n/n$ is a local property.*

3. We require the following lemmas for the proof of our theorems.

LEMMA 1. *If $\phi_1(t)$ is of bounded variation in $(0, \pi)$, then*

$$\frac{1}{n}\sum_{\nu=1}^{n} \nu A_\nu(x) = O(1),$$

as $n \to \infty$.

This lemma is a particular case of a more general result due to Prasad and Bhatt [**4**, Lemma 9].

LEMMA 2 [**1**, Lemmas 3 and 4]. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, then* λ_n *is non-negative and decreasing,* $n\Delta\lambda_n \to 0$, *and* $\lambda_n \log n = o(1)$, *as* $n \to \infty$.

LEMMA 3 [**3**; **4**, Lemma 3]. *If* $\{\lambda_n\}$ *is a convex sequence such that the series* $\sum n^{-1}\lambda_n$ *is convergent, then*

$$\sum_{n=1}^{m} \log (n+1)\Delta\lambda_n = O(1),$$

as $m \to \infty$.

LEMMA 4. *If* $P_n = 1 + 1/2 + \cdots + 1/(n+1)$, *then under the hypotheses of Theorem* 2

$$\sum_{\nu=0}^{[n/2]-2} \left| B_{\nu+1}\Delta\left\{\frac{(P_n - P_{n-\nu-1})\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right\}\right| = O\left(\frac{1}{n}\right).$$

PROOF OF THE LEMMA. We observe that

$$\begin{aligned}
&\sum_{\nu=0}^{[n/2]-2} \left| B_{\nu+1}\Delta\left\{\frac{(P_n - P_{n-\nu-1})\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right\}\right| \\
&\quad = O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{(P_{n-\nu-1} - P_{n-\nu-2})\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right] \\
&\qquad + O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{(P_n - P_{n-\nu-1})\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)^2}\right] \\
&\qquad + O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{(P_n - P_{n-\nu-1})\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^3(n-\nu)}\right] \\
&\qquad + O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{(P_n - P_{n-\nu-1})\log(\nu+2)\Delta\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right] \\
&\quad = O\left[\sum_{\nu=0}^{[n/2]-2} \frac{\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)^2}\right] + O\left[\sum_{\nu=0}^{[n/2]-2} \frac{\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right] \\
&\qquad + O\left[\sum_{\nu=0}^{[n/2]-2} \frac{\log(\nu+2)\Delta\lambda_{\nu+1}}{(\nu+1)(n-\nu)}\right] \\
&\quad = O\left[\frac{1}{n}\sum_{\nu=0}^{[n/2]-2} \frac{1}{n-\nu}\right] + O\left[\frac{1}{n}\sum_{\nu=0}^{[n/2]-2} \frac{1}{(\nu+1)^2}\right] \\
&\qquad + O\left[\frac{1}{n}\sum_{\nu=0}^{[n/2]-2} \log(\nu+2)\Delta\lambda_{\nu+1}\right] = O\left(\frac{1}{n}\right),
\end{aligned}$$

since for $0 \leqq \nu \leqq [n/2]$, $P_n - P_{n-\nu-1} = O(1)$, $\log(\nu+2)\lambda_{\nu+1} = O(1)$, by virtue of Lemma 2 and the series $\sum_{\nu=0}^{[n/2]-2} \log(\nu+2)\Delta\lambda_{\nu+1} = O(1)$, by Lemma 3.

This completes the proof of the lemma.

LEMMA 5. *If* $P_n = 1 + 1/2 + \cdots + 1/(n+1)$, *then under the hypotheses of Theorem* 2

$$\sum_{\nu=0}^{[n/2]-2} \left| B_{\nu+1}\Delta\left\{\frac{P_{n-\nu-1}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)}\right\}\right| = O\left[\frac{\log^2 n}{n}\right].$$

PROOF OF THE LEMMA. We observe that

$$\begin{aligned}
&\sum_{\nu=0}^{[n/2]-2} \left| B_{\nu+1}\Delta\left\{\frac{P_{n-\nu-1}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)}\right\}\right| \\
&= O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{(P_{n-\nu-1} - P_{n-\nu-2})\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)}\right] \\
&\quad + O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{P_{n-\nu-1}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right] \\
&\quad + O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{P_{n-\nu-1}\log(\nu+2)\Delta\lambda_{\nu+1}}{(\nu+1)(n-\nu)}\right] \\
&\quad + O\left[\sum_{\nu=0}^{[n/2]-2} |B_{\nu+1}| \frac{P_{n-\nu-1}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)^2}\right] \\
&= O\left[\sum_{\nu=0}^{[n/2]-2} \frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right] + O\left[\sum_{\nu=0}^{[n/2]-2} \frac{P_{n-\nu-1}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)}\right] \\
&\quad + O\left[\sum_{\nu=0}^{[n/2]-2} \frac{P_{n-\nu-1}}{n-\nu}\log(\nu+2)\Delta\lambda_{\nu+1}\right] \\
&\quad + O\left[\sum_{\nu=0}^{[n/2]-2} \frac{P_{n-\nu-1}}{(n-\nu)^2}\log(\nu+2)\lambda_{\nu+1}\right] \\
&= O\left[\frac{1}{n}\sum_{\nu=0}^{[n/2]-2} \frac{1}{n-\nu}\right] + O\left[\frac{\log^2 n}{n}\sum_{\nu=0}^{[n/2]-2} \frac{\lambda_{\nu+1}}{\nu+1}\right] \\
&\quad + O\left[\frac{\log n}{n}\sum_{\nu=0}^{[n/2]-2} \log(\nu+2)\Delta\lambda_{\nu+1}\right] + O\left[\frac{\log n}{n}\sum_{\nu=0}^{[n/2]-2} \frac{1}{n-\nu}\right] \\
&= O\left[\frac{\log^2 n}{n}\right],
\end{aligned}$$

by Lemma 3 and the hypothesis.

This completes the proof of the lemma.

LEMMA 6. *Under the same conditions as in Lemma* 5

$$\sum_{\nu=[n/2]}^{n-2} \left| B_{\nu+1}\Delta\left[\frac{\{(n+1)P_n-(n-\nu)P_{n-\nu-1}\}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right]\right|$$
$$= O(1)+O\left[\log n\sum_{\nu=1}^{n-1}\frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right].$$

PROOF OF THE LEMMA. We observe that

$$\sum_{\nu=[n/2]}^{n-2} \left| B_{\nu+1}\Delta\left[\frac{\{(n+1)P_n-(n-\nu)P_{n-\nu-1}\}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right]\right|$$
$$= O\left[\sum_{\nu=[n/2]}^{n-2}|B_{\nu+1}|\frac{\{(n-\nu)(P_{n-\nu-1}-P_{n-\nu-2})+P_{n-\nu-2}\}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right]$$
$$+O\left[\sum_{\nu=[n/2]}^{n-2}|B_{\nu+1}|\frac{\{(n+1)P_n-(n-\nu)P_{n-\nu-1}\}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^3(n-\nu)}\right]$$
$$+O\left[\sum_{\nu=[n/2]}^{n-2}|B_{\nu+1}|\frac{\{(n+1)P_n-(n-\nu)P_{n-\nu-1}\}\log(\nu+2)\Delta\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right]$$
$$+O\left[\sum_{\nu=[n/2]}^{n-2}|B_{\nu+1}|\frac{\{(n+1)P_n-(n-\nu)P_{n-\nu-1}\}\log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)^2}\right]$$
$$= O\left[\frac{\log n}{n}\sum_{\nu=[n/2]}^{n-2}\frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)}\right]+O\left[\log n\sum_{\nu=[n/2]}^{n-2}\frac{\log(\nu+2)\Delta\lambda_{\nu+1}}{(n-\nu)}\right]$$
$$+O\left[\log n\sum_{\nu=[n/2]}^{n-2}\frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right]$$
$$= O\left[\frac{\log n}{n}\sum_{\nu=[n/2]}^{n-2}\frac{1}{n-\nu}\right]+O\left[\frac{\log^3 n}{n}\right]$$
$$+O\left[\log n\sum_{\nu=1}^{n-1}\frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right]$$
$$= O(1)+O\left[\log n\sum_{\nu=1}^{n-1}\frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right],$$

since $n\Delta\lambda_n\to 0$, as $n\to\infty$, and $\log(\nu+2)\lambda_{\nu+1}=O(1)$, by virtue of Lemma 2.

4. By virtue of Lemma 1, Theorem 1 will be established if we prove Theorem 2.

PROOF OF THEOREM 2. Since

$$t_n = \sum_{\nu=0}^{n} P_{n-\nu}\, u_\nu / P_n,$$

where

$$u_n = \log(n+1)\lambda_n a_n / n,$$

we have

$$\begin{aligned} t_n - t_{n-1} &= \sum_{\nu=0}^{n-1}\left(\frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}}\right) u_{n-\nu} \\ &= \frac{1}{P_n \cdot P_{n-1}} \sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1}\right) u_{n-\nu} \\ &= \frac{1}{P_n \cdot P_{n-1}} \sum_{\nu=0}^{n-1}\left(\frac{P_n}{n-\nu} - \frac{P_{n-\nu-1}}{n+1}\right) u_{\nu+1}. \end{aligned}$$

Thus in order to establish the theorem we have to establish that

$$\begin{aligned} \sum_{n=2}^{\infty} \left| t_n - t_{n-1} \right| &= \sum_{n=2}^{\infty} \frac{(n+1)^{-1}}{P_n \cdot P_{n-1}} \\ &\quad \cdot \left| \sum_{\nu=0}^{n-1} \frac{\{(n+1)P_n - (n-\nu)P_{n-\nu-1}\} \log(\nu+2)\lambda_{\nu+1} a_{\nu+1}}{(\nu+1)(n-\nu)} \right| \\ &< \infty. \end{aligned} \tag{4.1}$$

Now let us consider

$$\begin{aligned} \sum_{1} &= \sum_{\nu=0}^{n-1} \frac{\{(n+1)P_n - (n-\nu)P_{n-\nu-1}\} \log(\nu+2)\lambda_{\nu+1} a_{\nu+1}}{(\nu+1)(n-\nu)} \\ &= \sum_{\nu=0}^{[n/2]-1} + \sum_{\nu=[n/2]}^{n-1} = \sum_{11} + \sum_{12}, \text{ say.} \end{aligned} \tag{4.2}$$

Also,

$$\begin{aligned} \sum_{11} &= \sum_{\nu=0}^{[n/2]-1} \frac{\{(n+1)P_n - (n-\nu)P_{n-\nu-1}\} \log(\nu+2)\lambda_{\nu+1} a_{\nu+1}}{(\nu+1)(n-\nu)} \\ &= (n+1) \sum_{\nu=0}^{[n/2]-1} \frac{(P_n - P_{n-\nu-1}) \log(\nu+2)\lambda_{\nu+1} a_{\nu+1}}{(\nu+1)(n-\nu)} \\ &\quad + \sum_{\nu=0}^{[n/2]-1} \frac{P_{n-\nu-1} \log(\nu+2)\lambda_{\nu+1} a_{\nu+1}}{(n-\nu)} \\ &= (n+1) \sum_{111} + \sum_{112}, \text{ say.} \end{aligned} \tag{4.3}$$

By Abel's transformation we have

$$
\begin{aligned}
\sum_{111} &= \sum_{\nu=0}^{[n/2]-1} \left\{\frac{(P_n - P_{n-\nu-1}) \log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right\} (\nu+1)a_{\nu+1} \\
&= \sum_{\nu=0}^{[n/2]-2} B_{\nu+1}\Delta \left\{\frac{(P_n - P_{n-\nu-1}) \log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right\} \\
&\quad + \frac{\{P_n - P_{(n-[n/2])}\} \log([n/2]+1)\lambda_{[n/2]}}{([n/2])^2(n-[n/2]+1)} B_{[n/2]} \\
&= O\left(\frac{1}{n}\right),
\end{aligned} \tag{4.4}
$$

by the application of Lemma 4.

Now applying Abel's transformation to $\sum_{112}$ we have

$$
\begin{aligned}
\sum_{112} &= \sum_{\nu=0}^{[n/2]-1} \frac{P_{n-\nu-1} \log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)} (\nu+1)a_{\nu+1} \\
&= \sum_{\nu=0}^{[n/2]-2} B_{\nu+1}\Delta \left\{\frac{P_{n-\nu-1} \log(\nu+2)\lambda_{\nu+1}}{(\nu+1)(n-\nu)}\right\} \\
&\quad + \frac{P_{(n-[n/2])} \log([n/2]+1)\lambda_{[n/2]}}{([n/2])(n-[n/2]+1)} B_{[n/2]} = O\left[\frac{\log^2 n}{n}\right], \text{ by Lemma 5.}
\end{aligned} \tag{4.5}
$$

Thus from (4.3), (4.4) and (4.5) we have

$$\sum_{11} = O(1). \tag{4.6}$$

Again by Abel's transformation

$$
\begin{aligned}
\sum_{12} &= \sum_{\nu=[n/2]}^{n-1} \frac{\{(n+1)P_n - (n-\nu)P_{n-\nu-1}\} \log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)} (\nu+1)a_{\nu+1} \\
&= \sum_{\nu=[n/2]}^{n-2} B_{\nu+1}\Delta \left[\frac{\{(n+1)P_n - (n-\nu)P_{n-\nu-1}\} \log(\nu+2)\lambda_{\nu+1}}{(\nu+1)^2(n-\nu)}\right] \\
&\quad - B_{[n/2]} \frac{(n+1)P_n - (n-[n/2])P_{(n-[n/2]-1)}}{([n/2]+1)^2(n-[n/2])} \\
&\quad \cdot \log([n/2]+2)\lambda_{[n/2]+1} \\
&\quad + B_n \frac{(n+1)P_n - P_0}{n^2} \log(n+1)\lambda_n \\
&= O(1) + O\left[\log n \sum_{\nu=1}^{n-1} \frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right] + O(\log^2 n\, \lambda_n), \text{ by Lemma 6.}
\end{aligned} \tag{4.7}
$$

Hence from (4.2), (4.6) and (4.7) we have

$$\sum_1 = O(1) + O\left[\log n \sum_{\nu=1}^{n-1} \frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right] + O[\log^2 n\, \lambda_n].$$

Therefore from (4.1) we have

$$\begin{aligned}
&\sum_{n=2}^{m} |t_n - t_{n-1}| \\
&= \sum_{n=2}^{m} \frac{(n+1)^{-1}}{P_n \cdot P_{n-1}}\left[O(1) + O(\log^2 n\, \lambda_n) + O(\log n) \sum_{\nu=1}^{n-1} \frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2}\right] \\
&= I_1 + I_2 + I_3, \text{ say.}
\end{aligned}$$

Clearly

$$I_1 = O(1),$$

and

$$I_2 = O(1),$$

as $m \to \infty$, by hypothesis.

Also

$$\begin{aligned}
I_3 &= O(1) \sum_{n=2}^{m} \frac{1}{n \log(n+1)} \sum_{\nu=1}^{n-1} \frac{\log(\nu+2)\lambda_{\nu+1}}{(n-\nu)^2} \\
&= O(1) \sum_{\nu=1}^{m-1} \log(\nu+2)\lambda_{\nu+1} \sum_{n=\nu+1}^{m} \frac{1}{n \log(n+1)(n-\nu)^2} \\
&= O(1) \sum_{\nu=1}^{m-1} \frac{\lambda_{\nu+1}}{\nu+1} \sum_{n=\nu+1}^{m} \frac{1}{(n-\nu)^2} \\
&= O(1) \sum_{\nu=1}^{m-1} \frac{\lambda_{\nu+1}}{\nu+1} \\
&= O(1),
\end{aligned}$$

as $m \to \infty$, under the hypothesis of the theorem.

Hence

$$\sum_{n=2}^{\infty} |t_n - t_{n-1}| < \infty,$$

which demonstrates the truth of (4.1).

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ON THE COMPLETION OF TRACTABLE NORMED ALGEBRAS

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A normed algebra B (norm $\|\cdot\|$; we always assume that B is commutative and possesses a multiplicative unit) is said to be tractable if the intersection of its $\|\cdot\|$-closed maximal ideals is trivial (cf. [1]). If $\overline{B}$ denotes the completion of B with respect to $\|\cdot\|$, then we might ask if $\overline{B}$ can have a nontrivial radical. Examples have been given which show that this is indeed the case. In this note, we give a class of tractable normed algebras whose completions have nontrivial radicals and further show that it is possible to do this for certain algebras of functions which are complete in sup norm (but not all continuous functions on their carrier spaces). We make use of algebraic extensions of normed algebras and need the following results.

Lemma 1. *If A is a normed algebra (norm $\|\cdot\|$) and if $a\in A$, then $A[x]/(x^2-a)$ is a normed algebra under*

$$\|a_0 + a_1 + (x^2 - a)\|_1 = \|a_0\| + \|a_1\|\,\|a\|.$$

Lemma 2. *If A is a tractable normed algebra and if $a\in A$, then $A[x]/(x^2-a)$ is a tractable normed algebra under $\|\cdot\|_1$ if and only if a is not a zero divisor in A.*

Lemma 3. *The $\|\cdot\|_1$-completion of $A[x]/(x^2-a)$, $a\in A$, is (isomorphic to) $\overline{A}[x]/(x^2-a)$, where $\overline{A}$ denotes the $\|\cdot\|$-completion of A.*

Lemmas 1 and 2 have been proved by Arens and Hoffman [1]. The

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