ON MATRICES OVER THE RING OF CONTINUOUS COMPLEX VALUED FUNCTIONS ON A STONIAN SPACE

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1. **Introduction.** The purpose of this note is to prove three theorems concerning matrices with entries from the algebra $C(\mathfrak{X})$ of continuous complex valued functions on an extremely disconnected, compact Hausdorff space $\mathfrak{X}$. (Such spaces are sometimes called Stonian after M. H. Stone, who considered them in [**11**]. They arise naturally as the maximal ideal spaces of abelian AW^*-algebras [**10**].) We first prove that any monic polynomial equation with coefficients in $C(\mathfrak{X})$ has a solution in the coefficient algebra. We then use this result to prove that any matrix with entries from $C(\mathfrak{X})$ is unitarily equivalent to another such matrix which is in upper triangular form. Finally, this allows us to prove that any invertible matrix of the above form has roots of all orders in the algebra of all $n\times n$ matrices over $C(\mathfrak{X})$. In writing this note the authors were motivated by several considerations. In the first place, Theorem 1 is (in disguise) a generalization with a new proof of a known result for abelian W^*-algebras. Theorem 2, in turn, can be regarded as a fragment of structure theory for certain operator algebras, and is used extensively in [**9**]. Finally, Theorem 3 generalizes the known result that invertible $n\times n$ complex matrices have roots of all orders, and thereby provides a large class of invertible operators on Hilbert space which also have this property.

2. Throughout this paper $\mathfrak{X}$ will denote an extremely disconnected, compact Hausdorff space, and $C(\mathfrak{X})$ will denote the C^*-algebra of continuous complex valued functions on $\mathfrak{X}$ under the sup norm. We first prove a lemma which allows us to construct functions in $C(\mathfrak{X})$ by a "localizing" procedure.

LEMMA 2.1. *Suppose that $\{\mathfrak{U}_i\}_{i\in I}$ is a disjoint collection of compact open subsets of $\mathfrak{X}$, and suppose that for $i\in I$, f_i is a continuous complex valued function defined on $\mathfrak{U}_i$. If the f_i are uniformly bounded in modulus, then there is a function f defined and continuous on $\mathfrak{U}=[\bigcup_{i\in I}\mathfrak{U}_i]^{cl}$ such that f extends each f_i.*

PROOF. Stone proved in [**11**] that the algebra of all real continuous functions on $\mathfrak{X}$ is a boundedly complete lattice, and the result follows

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by considering the real and imaginary parts of the f_i.

The following lemma was pointed out to us by G. R. MacLane.

LEMMA 2.2. *Suppose* $\mathfrak{I}$ *is any topological space, and* $P(t, z) = z^n + a_{n-1}(t)z^{n-1} + \cdots + a_0(t)$, *where the* $a_i \in C(\mathfrak{X})$. *If* z_0 *is a root of multiplicity* μ *of the polynomial equation* $P(t_0, z) = 0$, *and if* $\delta > 0$ *is such that* $P(t_0, z)$ *has no root other than* z_0 *satisfying* $|z - z_0| < \delta$, *then there is a neighborhood* $\mathfrak{N}(t_0, \delta)$ *of* t_0 *such that for* $t \in \mathfrak{N}(t_0, \delta)$, $P(t, z) = 0$ *has exactly* μ *roots (counting multiplicities) satisfying* $|z - z_0| < \delta$.

PROOF. Let $m = \min_{|z - z_0| = \delta} |P(t_0, z)|$. It follows from Lemma 5F of [8] that there exists $\mathfrak{N}(t_0, \delta)$ of t_0 such that $|P(t, z) - P(t_0, z)| < m$ whenever $t \in \mathfrak{N}(t_0, \delta)$ and $|z - z_0| = \delta$. The proof is completed by applying Rouché's Theorem to $P(t, z) = P(t_0, z) + \{P(t, z) - P(t_0, z)\}$. (See [1] for a statement of Rouché's Theorem.)

Using these lemmas we now prove

THEOREM 1. *If* $\mathfrak{X}$ *is a Stonian space,* $a_0, \cdots, a_{n-1} \in C(\mathfrak{X})$ *and* $P(t, z)$ *is as in Lemma* 2.2, *then there is a function* $f \in C(\mathfrak{X})$ *satisfying* $P(t, f(t)) \equiv 0$.

PROOF. Consider collections $\{\mathfrak{U}_i\}$ of disjoint nonempty compact open subsets $\mathfrak{U}_i$ of $\mathfrak{X}$ with the property that for each $\mathfrak{U}_i \in \{\mathfrak{U}_i\}$, there is a function $f_i \in C(\mathfrak{U}_i)$ such that $P(t, f_i(t)) \equiv 0$ for $t \in \mathfrak{U}_i$, and let $\{\mathfrak{U}_i\}_{i \in I}$ be a maximal collection of such $\mathfrak{U}_i$. Since $\mathfrak{X}$ is compact, the a_k are uniformly bounded, and it is not hard to see that this guarantees that the f_i are also uniformly bounded. Thus, in view of Lemma 2.1, it suffices to show that $\mathfrak{U} = [\bigcup_{i \in I} \mathfrak{U}_i]^{cl}$ is equal to $\mathfrak{X}$, and we do this as follows. Suppose the compact open set $\mathfrak{X} - \mathfrak{U}$ is not void, and for $t \in \mathfrak{X} - \mathfrak{U}$ denote by $\mu(t)$ the multiplicity of a root of minimum multiplicity of $P(t, z) = 0$. Choose $t_0 \in \mathfrak{X} - \mathfrak{U}$ so that $\mu(t_0) \leqq \mu(t)$ for all $t \in \mathfrak{X} - \mathfrak{U}$, and let z_0 be a root of $P(t_0, z) = 0$ with multiplicity $\mu(t_0)$. By Lemma 2.2, there is a $\delta > 0$ and a compact open neighborhood $\mathfrak{N}(t_0, \delta) \subset \mathfrak{X} - \mathfrak{U}$ of t_0 such that $t \in \mathfrak{N}(t_0, \delta)$ implies that $P(t, z) = 0$ has exactly $\mu(t_0)$ roots (counting multiplicities) satisfying $|z - z_0| < \delta$. Since $\mu(t_0)$ is minimal, these roots must coincide, and the function $f(t)$ defined on $\mathfrak{N}(t_0, \delta)$ to be this unique root satisfies $P(t, f(t)) \equiv 0$ on $\mathfrak{N}(t_0, \delta)$. We complete the argument by showing that f is continuous on $\mathfrak{N}(t_0, \delta)$, thus contradicting the maximality of $\{\mathfrak{U}_i\}_{i \in I}$. Let $t_1 \in \mathfrak{N}(t_0, \delta)$ and suppose ϵ satisfies $0 < \epsilon < \delta - |z_0 - f(t_1)|$. By Lemma 2.2, there is a neighborhood $\mathfrak{N}(t_1, \epsilon) \subset \mathfrak{N}(t_0, \delta)$ of t_1 such that for $t \in \mathfrak{N}(t_1, \epsilon)$, the equation $P(t, z) = 0$ has exactly μ roots satisfying $|z - f(t_1)| < \epsilon$. These roots clearly must coincide and be equal to $f(t)$, so that one has $|f(t) - f(t_1)| < \epsilon$ for $t \in \mathfrak{N}(t_1, \epsilon)$.

Corollary 2.3. *Suppose* $\mathbf{A}$ *is any abelian* AW^**-algebra containing the elements* $A_0, \cdots, A_{n-1}$. *Then the polynomial equation* $Z^n+A_{n-1}Z^{n-1}+\cdots+A_0=0$ *has a solution in* $\mathbf{A}$.

3. **Triangularization.** We denote by M_n the full ring of $n\times n$ complex matrices under the operator norm. Consider the *-algebra $M_n(\mathfrak{X})$ of all continuous functions from $\mathfrak{X}$ to M_n where the algebraic operations in $M_n(\mathfrak{X})$ are defined pointwise. If $A\in M_n(\mathfrak{X})$, then setting $\|A\|=\sup_{t\in\mathfrak{X}}\|A(t)\|$ turns $M_n(\mathfrak{X})$ into a C^*-algebra, and it is clear that $M_n(\mathfrak{X})$ can be identified with the C^*-algebra of all $n\times n$ matrices with entries from the abelian algebra $C(\mathfrak{X})$. (Thus $M_n(\mathfrak{X})$ is in fact an n-homogeneous AW^*-algebra [7].) The following theorem is a useful tool in the study of certain operator algebras.

Theorem 2. *If* $A\in M_n(\mathfrak{X})$, *then there is a unitary element* $U\in M_n(\mathfrak{X})$ *such that* $B=U^*AU$ *has the property that* $B(t)$ *is in upper triangular form for each* $t\in\mathfrak{X}$.

In order to prove the theorem, we first develop some preliminary results. Let E_n denote the n-dimensional vector space of all n-tuples of complex numbers under the Euclidean norm, and let $E_n(\mathfrak{X})$ be the collection of all continuous functions from $\mathfrak{X}$ to E_n. (Clearly $E_n(\mathfrak{X})$ coincides with the collection of all n-tuples of elements of $C(\mathfrak{X})$.)

Lemma 3.1. *If* $A\in M_n(\mathfrak{X})$, *then there exist* $\lambda\in C(\mathfrak{X})$ *and* $v\in E_n(\mathfrak{X})$ *such that for each* $t\in\mathfrak{X}$, $A(t)v(t)=\lambda(t)v(t)$ *and* $\|v(t)\|=1$.

Proof. Theorem 1 guarantees the existence of a continuous solution $\lambda(t)$ of the characteristic equation $\det[A(t)-\lambda I]=0$. Now consider collections $\{\mathfrak{U}_i\}$ of disjoint nonempty compact open subsets of $\mathfrak{X}$ with the property that if $\mathfrak{U}_i\in\{\mathfrak{U}_i\}$, then there exists a $v_i\in E_n(\mathfrak{U}_i)$ satisfying $A(t)v_i(t)=\lambda(t)v_i(t)$ and $\|v_i(t)\|=1$ for each $t\in\mathfrak{U}_i$. Choose a maximal collection $\{\mathfrak{U}_i\}_{i\in I}$ by Zorn, and let $\mathfrak{U}=[\bigcup_{i\in I}\mathfrak{U}_i]^{cl}$. Again it is clear (in view of Lemma 2.1) that it suffices to prove $\mathfrak{U}=\mathfrak{X}$. If $\mathfrak{X}-\mathfrak{U}\neq\varnothing$, choose $t_0\in\mathfrak{X}-\mathfrak{U}$ such that the rank $r(t)$ of the matrix $A(t)-\lambda(t)I$ assumes its maximum r_0 at t_0. If $r_0=0$, then $A(t)\equiv\lambda(t)I$ on $\mathfrak{X}-\mathfrak{U}$, and clearly this contradicts the maximality of the collection $\{\mathfrak{U}_i\}_{i\in I}$. Thus we take $r_0>0$ and consider a nonzero $r_0\times r_0$ minor M of $\det[A(t_0)-\lambda(t_0)I]$. There is a compact open neighborhood $\mathfrak{N}\subset\mathfrak{X}-\mathfrak{U}$ of t_0 such that the same minor M of $\det[A(t)-\lambda(t)I]$ remains a nonzero minor of maximum size on $\mathfrak{N}$. To obtain a vector $v\in E_n(\mathfrak{N})$ satisfying $A(t)v(t)=\lambda(t)v(t)$, $\|v(t)\|=1$, and thus arrive at a contradiction, we proceed as follows. Make any continuous nonzero choice for the appropriate $n-r_0$ components of v on $\mathfrak{N}$, apply Kramer's rule to the system of equations $[A(t)-\lambda(t)I]v(t)=0$, and normalize the result.

LEMMA 3.2. *If $A \in M_n(\mathfrak{X})$, then there is a unitary element $U \in M_n(\mathfrak{X})$ such that $B = (b_{ij}) = U^*AU$ has the property that $b_{i1}(t) \equiv 0$ for $i=2, \cdots, n$.*

PROOF. If we consider, as before, disjoint collections $\{\mathfrak{U}_i\}$ of nonempty compact open subsets on which the result is true, we can reduce the problem to constructing an appropriate unitary valued function U on some compact open neighborhood of an arbitrary point $t_0 \in \mathfrak{X}$, and this we do as follows. Let $v \in E_n(\mathfrak{X})$ be the function whose existence is established in Lemma 3.1, and extend the vector $v(t_0)$ to an orthonormal basis $v(t_0) = v_1(t_0), v_2(t_0), \cdots, v_n(t_0)$ of E_n. Let $\mathfrak{N}$ be a compact open neighborhood of t_0 such that for $t \in \mathfrak{N}$ the vectors $v(t) = v_1(t), v_2(t_0), \cdots, v_n(t_0)$ remain linearly independent. If for $t \in \mathfrak{N}$, the Gram-Schmidt process (including normalization) is applied to the above basis for E_n, there will result n functions $v_i(t) \in E_n(\mathfrak{N})$ such that for $t \in \mathfrak{N}$ the $v_i(t)$ form an orthonormal basis for E_n. Now define a unitary valued function $U \in M_n(\mathfrak{N})$ by defining, for $t \in \mathfrak{N}$, $U(t)$ to be the matrix whose ith column is the vector $v_i(t)$. Recalling that $v_1(t)$ is always an eigenvector for $A(t)$ corresponding to the eigenvalue $\lambda(t)$, an easy calculation shows that for $t \in \mathfrak{N}$, $B(t) = U^*(t)A(t)U(t)$ has the required form, and this completes the argument.

Lemma 3.2 allows us to begin the triangularization process on an arbitrary element $A \in M_n(\mathfrak{X})$. The proof of Theorem 2 can now be completed by an obvious induction argument on n, and we omit further details of that argument. We do record an obvious corollary of Theorem 2 for future reference.

COROLLARY 3.3. *If A is any hermitian element in $M_n(\mathfrak{X})$, then there is a unitary $U \in M_n(\mathfrak{X})$ such that $U^*AU(t)$ is diagonal for each $t \in \mathfrak{X}$.*

4. ***K*th roots.** With the help of Theorem 2 we begin our attack on

THEOREM 3. *If A is an invertible element in $M_n(\mathfrak{X})$, then every equation of the form $Z^k = A$ where k is a positive integer has a solution in $M_n(\mathfrak{X})$.*

PROOF. We argue by induction on n. The case $n=1$ is taken care of by an application of Theorem 1. If $n>1$, we can suppose, as a result of Theorem 2, that $A = (a_{ij})$ is in upper triangular form. Since A is invertible, the product $\prod_{i=1}^n a_{ii}(t)$ never vanishes, and this in turn tells us that $A_1 \in M_{n-1}(\mathfrak{X})$ defined by deleting the nth row and nth column of A is invertible in $M_{n-1}(\mathfrak{X})$. By applying the induction hypothesis, we can assume that there is an element $B_1 = (b_{ij}) \in M_{n-1}(\mathfrak{X})$ in upper triangular form and satisfying $B_1^k = A_1$. For reasons which will be clear shortly, we also include in the induction hypothesis the assumption:

(*) For any i, j satisfying $1 \leqq i, j \leqq n-1$, and any $t \in \mathfrak{X}$, $[b_{ii}(t)]^k = [b_{jj}(t)]^k$ if and only if $b_{ii}(t) = b_{jj}(t)$.

Let B be the matrix

$$\begin{pmatrix} B_1 & \vdots & \mu \\ \cdots & \cdots & \cdots \\ 0 & \vdots & b_{nn} \end{pmatrix}$$

where $\mu \in E_{n-1}(\mathfrak{X})$ and $b_{nn} \in C(\mathfrak{X})$ are to be determined so that $B^k = A$. A calculation shows that B^k is the matrix

$$\begin{pmatrix} A_1 & \vdots & \gamma \\ \cdots & \cdots & \cdots \\ 0 & \vdots & b_{nn}^k \end{pmatrix},$$

where

$$(\Delta) \qquad \gamma = (B_1^{k-1} + b_{nn}B_1^{k-2} + \cdots + b_{nn}^{k-1} I)\mu$$

and the expression in the parentheses is to be regarded as a matrix operating on the vector μ. The following lemma ensures that b_{nn} can be chosen to satisfy the induction hypothesis.

LEMMA 4.1. *Suppose that $b_{11}, b_{22}, \cdots, b_{n-1,n-1}, a_{nn}$ are nonvanishing functions in $C(\mathfrak{X})$, and suppose that the b_{ii} satisfy (*). Then there is a $b_{nn} \in C(\mathfrak{X})$ such that $b_{nn}^k \equiv a_{nn}$ and such that the enlarged set $\{b_{ii}\}$ satisfies the relation (**) obtained from (*) by substituting n for $n-1$.*

PROOF. By Theorem 1, there exists $b \in C(\mathfrak{X})$ satisfying $b^k \equiv a_{nn}$. For $1 \leqq r \leqq k$, define the closed set $\mathfrak{F}_r \subset \mathfrak{X}$ by: $t \in \mathfrak{F}_r$ if and only if $b(t) = e^{i2\pi r/k} b_{jj}(t)$ for some j satisfying $1 \leqq j \leqq n-1$. By virtue of (*), the $\mathfrak{F}_r$ are mutually disjoint, and since $\mathfrak{X}$ is normal and extremely disconnected, there are mutually disjoint compact open sets $\mathfrak{U}_1, \cdots, \mathfrak{U}_k \subset \mathfrak{X}$ such that for $1 \leqq r \leqq k$, $\mathfrak{F}_r \subset \mathfrak{U}_r$. Now define the function $b_{nn} \in C(\mathfrak{X})$ by setting $b_{nn}(t) = e^{-i2\pi r/k} b(t)$ for $t \in \mathfrak{U}_r$, $1 \leqq r \leqq k$, and $b_{nn}(t) = b(t)$ for $t \in \mathfrak{X} - \bigcup \mathfrak{U}_r$. This function clearly satisfies the requirements of the lemma.

Continuing with the proof of the theorem, choose b_{nn} as in the above lemma, and observe that to complete the solution of $B^k = A$, it suffices to solve the equation (Δ) for $\mu \in E_{n-1}(\mathfrak{X})$, where γ is the column vector $(a_{1n}, \cdots, a_{n-1,n})$. It is clear that in order for the expression in the parentheses in (Δ) to be an invertible element of $M_{n-1}(\mathfrak{X})$, it is sufficient that $B_1(t)^{k-1}+b_{nn}(t)B_1(t)^{k-2}+ \cdots +b_{nn}(t)^{k-1}I$ be an invertible matrix for each $t \in \mathfrak{X}$. That this is true follows from the fact that the eigenvalues of the above matrix are exactly the numbers $b_{ii}(t)^{k-1}+b_{nn}(t)b_{ii}(t)^{k-2}+ \cdots +b_{nn}(t)^{k-1}$, $1 \leqq i \leqq n-1$, and by virtue of (**) none of these numbers is zero. Thus $\mu \in E_{n-1}(\mathfrak{X})$ is determined by (Δ), and the proof of the theorem is complete.

COROLLARY 4.2. *Suppose R is a finite AW^*-algebra of type I, and m is a positive integer such that R has no n-homogeneous component with $n>m$. Let A be any invertible element in R. Then every equation of the form $Z^k=A$ where k is a positive integer has a solution in R.*

PROOF. One knows from [6] that R is a C^*-sum of AW^*-algebras $\{R_i\}_{i\in J}$ where each R_i is i-homogeneous and J is some subset of the first m positive integers. Since A is invertible in R, the component A_i of A in R_i is invertible in R_i, and thus it suffices to prove the corollary in the case that R is n-homogeneous. Let C be the center of the n-homogeneous algebra R, and let $\mathfrak{X}$ be the maximal ideal space of C. Then $\mathfrak{X}$ is an extremely disconnected compact Hausdorff space, and it follows from standard matrix unit arguments that R is C^*-isomorphic to the AW^*-algebra $M_n(\mathfrak{X})$. The result follows from Theorem 3.

5. Remarks.

(1) The truth of Corollary 2.3 for coefficients in an abelian W^*-algebra is a consequence of Theorem 2 of [4] and Theorem 1, page 117, of [3].

(2) In the related note [2], the authors consider any finite AW^*-algebra R of type I having n_k-homogeneous components where $n_k \to \infty$ and exhibit an invertible element $A \in R$ such that A has no square root in R. Thus the limitation in Corollary 4.2 that the n-homogeneous components be such that n is bounded away from ∞ is necessary.

(3) Hille and Phillips have shown [5, Theorem 9.5.4] that a sufficient condition for an operator A to have roots of all orders in the Banach algebra generated by A and the identity is that zero belong to the unbounded component of the resolvent set of A.

(4) It is easy to see that the scarcity of compact open subsets in a space $\mathfrak{Z}$ can cause the analogs of the above three theorems to be false. For example, if $\mathfrak{Z}$ is the closed unit disc in the complex plane

and $a(t)=t$, then Theorem 1 fails for the equation $z^2-a(t)=0$ and Theorem 2 fails for the matrix

$$\begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}.$$

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