ON EXTREME AVERAGING OPERATORS
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1. Introduction. With $X$ a compact Hausdorff space, let $C(X)$ denote the Banach algebra of continuous scalar valued functions on $X$, with scalars either the real or the complex numbers, and with the usual pointwise operations and uniform norm. Let $\mathfrak{a}$ be a closed subalgebra of $C(X)$ containing constant functions, and complex conjugate functions in the complex case. We are concerned with the set $\varTheta(\mathfrak{a})$ of all positive bounded projections in $C(X)$ onto fixed range space $\mathfrak{a}$.

It has been shown by Birkhoff [1] and others [2; 3] that the members of $\varTheta(\mathfrak{a})$ are averaging operators, in the sense $\mu(fg) = (\mu f)(\mu g)$, $f, g \in C(X)$, with structure of the following type. Subalgebras $\mathfrak{a}$ of the kind specified correspond 1-1 to upper semicontinuous decompositions of $X$ into closed equivalence classes, the equivalence classes for a given $\mathfrak{a}$ consisting of the sets whose points are not separated by the functions in $\mathfrak{a}$. A given $\mu \in \varTheta(\mathfrak{a})$ determines for each equivalence class a regular Borel probability measure living on the equivalence class and for each $f \in C(X)$ the value of $\mu f$ on an equivalence class is obtained by averaging $f$ over the equivalence class with respect to the associated measure.

As it turns out, $\varTheta(\mathfrak{a})$ is a bounded convex set and we are interested in the extreme point properties of this convex set. Generalizing a result of Davis [4], we characterize the extreme points of $\varTheta(\mathfrak{a})$ as those positive projections which are algebraic homomorphisms of $C(X)$ onto $\mathfrak{a}$. Examples are given where (i) $\varTheta(\mathfrak{a})$ is empty (ii) non-empty $\varTheta(\mathfrak{a})$ has no extreme points (iii) nonempty $\varTheta(\mathfrak{a})$ has a few extreme points but is larger than their convex hull (iv) nonempty $\varTheta(\mathfrak{a})$ is the closed convex hull of its extreme points. Situation (iv) obtains, in particular, when $X$ is the extremally disconnected Stone space of a finite measure algebra.

2. Preliminaries. Let $\phi$ denote the equivalence in $X$ associated with $\mathfrak{a}$. That is, $x_1 \phi x_2$ if and only if $g(x_1) = g(x_2)$ for all $g \in \mathfrak{a}$. The quotient space $Y = X/\phi$ of equivalence classes is compact Hausdorff and the canonical continuous mapping $\pi: X \to Y$ is closed. The subalgebra $\mathfrak{a}$ is isometrically algebraically isomorphic to $C(Y)$ under the 1-1 correspondence $\rho: C(Y) \to \mathfrak{a}$ defined by

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We represent the conjugate space $C^*(X)$ of bounded linear functionals on $C(X)$ as the space of regular bounded scalar valued (signed) Borel measures on $X$, with total variation norm. We denote by $X$ the $\sigma$-field of Borel subsets of $X$. We denote by $(f, \lambda)$ the value $(f, \lambda) = \int f(x)\lambda(dx)$ of $\lambda \in C^*(X)$ at $f \in C(X)$.

Proof of the following representation theory for members of $\mathcal{P}(\mathfrak{a})$ appears in [1; 2; 3]; we use the notation of Wright [3].

**Theorem 1.** The members of $\mathcal{P}(\mathfrak{a})$ correspond 1-1 to certain $C^*(X)$ valued functions on $Y$, as follows. If $\sigma: Y \rightarrow C^*(X)$ corresponds to $\mu \in \mathcal{P}(\mathfrak{a})$ then $\sigma$ and $\mu$ are related by (i), and $\sigma$ has properties (ii)–(iv):

(i) $\mu f(x) = (f, \sigma_x) = \int f(x')\sigma_{x'}(dx'), \ x \in X, f \in C(X),$

(ii) $\sigma: Y \rightarrow C^*(X)$ is continuous with the $C(X)$ (weak*) topology in $C^*(X),$

(iii) $\sigma_y \geq 0, \sigma_y(X) = 1, y \in Y,$

(iv) $\sigma_y(E) = \sigma_{y}(E \cap \pi^{-1}y), E \subseteq X, y \in Y.$

3. Extreme points. A cross section of the mapping $\pi: X \rightarrow Y$ is any continuous function $\gamma: Y \rightarrow X$ such that $\pi\gamma$ is the identity mapping on $Y$. When the measure $\sigma_y$ of (2) is concentrated at a single point $\gamma(y) \in \pi^{-1}y$ for each $y \in Y$ then $\gamma: Y \rightarrow X$ is a cross section of $\pi$, from (2.ii) and (2.iii). The corresponding projection is an algebraic homomorphism of $C(X)$ onto $\mathfrak{a}$; in fact, we have (as noted by Wright [3]):

**Theorem 2.** The members of $\mathcal{P}(\mathfrak{a})$ which are homomorphisms of $C(X)$ onto $\mathfrak{a}$ correspond 1-1 to cross sections of $\pi: X \rightarrow Y$.

The connection with extreme points is given by

**Theorem 3.** A positive projection $\mu \in \mathcal{P}(\mathfrak{a})$ is an extreme point of $\mathcal{P}(\mathfrak{a})$ if and only if $\mu$ is an algebraic homomorphism of $C(X)$ onto $\mathfrak{a}$.

**Proof.** The if part of the statement is due to Wright [3]; we reproduce the proof for the sake of completeness. Suppose $\mu \in \mathcal{P}(\mathfrak{a})$ is a homomorphism. In representation (2), this becomes $(fg, \sigma_y) = (f, \sigma_y)(g, \sigma_y), y \in Y, f, g \in C(X)$, which is to say that for each $y \in Y$ the functional $\sigma_y \in C^*(X)$ is a homomorphism of $C(X)$ into the scalars, and onto because (2.iii) implies $(1, \sigma_y) = 1$. It is well known that $\sigma_y$ must then be for each $y$ a unit point measure, that is, an extreme point of the set of regular Borel probability measures on $X$. It follows at once that $\mu$ is necessarily an extreme point of $\mathcal{P}(\mathfrak{a})$. Conversely, suppose $\mu$ is an extreme point of $\mathcal{P}(\mathfrak{a})$. We will show that
the corresponding \( \sigma \) has the property that for each \( y \in Y \) and \( E \subseteq X \), either \( \sigma_y(E) = 0 \) or \( \sigma_y(E) = 1 \). It will then follow from well-known arguments that \( \sigma_y \) is for each \( y \) a homomorphism of \( C(X) \) onto the scalars, and hence \( \mu \) a homomorphism onto \( \mathfrak{A} \). Suppose to the contrary that \( 0 < \sigma_y(E_0) < 1 \) obtains for some \( y_0 \in Y, E_0 \subseteq X \). Since \( \sigma_{y_0} \) is a regular Borel probability measure, there exist closed sets \( F_1 \subseteq E_0, F_2 \subseteq E_0 \), such that \( \sigma_{y_0}(F_1) > 0, \sigma_{y_0}(F_2) > 0 \). Because \( X \) is normal, there exists \( f_1 \in C(X) \) such that \( f_1(F_1) = 1, f_1(F_2) = 0, 0 \leq f_1 \leq 1 \). With \( f_2 = 1 - f_1 \), define \( \tau^{(i)} : Y \to C^*(X) \) by

\[
\tau^{(i)}(E) = \int_E f_i(x) \sigma_y(dx), \quad y \in Y, \ E \subseteq X, \ i = 1, 2,
\]

noting that \( \tau^{(1)} + \tau^{(2)} = \sigma \). The measure \( \tau^{(i)}_y \) lives on \( \pi^{-1}y \), clearly, and from \( (f, \tau^{(i)}_y) = (f f_i, \sigma_y) \), \( y \in Y, f \in C(X) \) (Radon-Nikodym) it follows that each \( \tau^{(i)} \) induces a positive bounded linear mapping of \( C(X) \) into \( C(Y) \).

In particular, the nonnegative real functions \( \tau^{(0)}_y(X) = (1, \tau^{(0)}_y) = (f_i, \sigma_y) \) are continuous on \( Y \), and moreover, \( \tau^{(0)}_y(X) \geq \sigma_y(F_i) > 0, \ i = 1, 2 \). For an arbitrary \( a > 0 \) there exists then an open neighborhood \( V \) of \( y_0 \) such that

\[
| \tau^{(i)}_y(X) - \tau^{(0)}_y(X) | < a \tau^{(0)}_y(X), \quad i = 1, 2, \ y \in V.
\]

Because \( Y \) is completely regular, there exists \( h \in C(Y) \) such that \( h(y_0) = 1, h(V') = 0, 0 \leq h \leq 1 \). With \( b \) to be chosen presently, define

\[
\sigma_y^{(i)}(E) = \left[ \frac{bh(y)}{\tau^{(0)}_y(X)} \right] \tau^{(i)}_y(E) + \left[ 1 - \frac{bh(y)}{\tau^{(0)}_y(X)} \right] \sigma_y(E), \quad y \in Y, \ E \subseteq X, \ i = 1, 2.
\]

Now with \( 0 < b < (a + 1)^{-1} \), the continuous coefficients in square brackets above are nonnegative real on \( Y \), and it is readily verified that each \( \sigma^{(i)} \) induces a positive projection in \( C(X) \) onto \( \mathfrak{A} \). A straightforward calculation gives

\[
(f_1, \sigma_y^{(1)}) - (f_1, \sigma_y^{(2)}) = b \frac{(f_1, \sigma_y^{2}) - (f_1, \sigma_y^{1})^2}{(f_1, \sigma_y)(f_2, \sigma_y)},
\]

and since \( f_1 \neq \text{const} \) almost everywhere \([\sigma_y])\), the above difference does not vanish; that is, \( \sigma^{(1)} \) and \( \sigma^{(2)} \) correspond to distinct means. Finally, the identity \( \sigma = \tau^{(0)}_y(X)\sigma^{(1)} + \tau^{(2)}_y(X)\sigma^{(2)} \) shows that \( \sigma \) does not correspond to an extreme mean, contrary to hypothesis. Hence it must be the case that \( \sigma \) takes values 0 and 1 only, implying that
each $\sigma_\tau$ is a homomorphism. (Proof as in [5, p. 444]; we omit the details.)

**Corollary.** Extreme points of $\Theta(\alpha)$ correspond 1-1 to cross sections of $\pi: X \to Y$.

4. **Some examples.** The author is indebted to E. A. Michael for Example 1 and for a suggestion which led to Examples 2 and 3.

**Example 1.** $\Theta(\alpha)$ is empty. Let $X$ be the interval $[0, 3]$ and let $\alpha$ consist of the functions in $C(X)$ which are constant on the subinterval $[1, 2]$. The equivalence classes consist of the interval $[1, 2]$ and the remaining individual points of $X$. From Theorem 1, $\mu \in \Theta(\alpha)$ would have the form

$$\mu f(x) = \int_1^2 f(x') \sigma(dx'), x \in [1, 2]; \mu f(x) = f(x), x \in X - [1, 2].$$

For no measure $\sigma$ is $\mu f$ continuous if $f(1) \neq f(2)$, and it follows that $\Theta(\alpha)$ is empty.

**Example 2.** Nonempty $\Theta(\alpha)$ has no extreme points. Let $X$ be the interval $[0, 2]$ with endpoints identified, i.e., a circle, and let $\alpha$ consist of the functions $g$ in $C(X)$ satisfying $g(x) = g(x + 1), 0 \leq x < 1$. The equivalence classes are the pairs $\{x, x + 1\}, 0 \leq x < 1$, and $Y$ may be represented as the interval $[0, 1]$ with endpoints identified. Let $s(x), 0 \leq x \leq 2$, be any continuous function satisfying $0 \leq s(x) \leq 1, 0 \leq x \leq 2$, and $s(x) + s(x + 1) = 1, 0 \leq x \leq 1$. Then $\mu: C(X) \to \alpha$ defined by $\mu f(x) = \mu f(x + 1) = s(x)f(x) + s(x + 1)f(x + 1), 0 \leq x \leq 1$, is a positive bounded projection onto $\alpha$, as is easily verified. It is not hard to see that the canonical projection $\pi: X \to Y$ in this example has no cross sections. It follows from Theorem 3 corollary that $\Theta(\alpha)$ has no extreme points.

The mapping $\pi$ is open in Example 2, where projections exist, and not open in Example 1, where projections do not exist, and Michael [6] has shown that sufficient conditions for $\Theta(\alpha)$ to be nonempty are that $\pi$ be open and $X$ be metric. Openness of $\pi$ is not necessary for the existence of projections, however, as the next example shows.

**Example 3.** Nonempty $\Theta(\alpha)$ has no extreme points; $\pi$ is not open. Let $X$ be the interval $[0, 2]$, but now without identifications. Let $\alpha$ consist of the functions $g$ in $C(X)$ satisfying $g(x) = g(x + 1), 0 \leq x \leq 1$. The equivalence classes are the pairs $\{x, x + 1\}, 0 < x < 1$, and the triple $\{0, 1, 2\}$. Again, $Y$ is a circle; $\pi$ is no longer open. Projections onto $\alpha$ exist and have the same form as in Example 2, except that $s$ must now also satisfy $s(0) = s(2) = 0$. It is still the case that $\pi$ has no cross sections, whence $\Theta(\alpha)$ has no extreme points.
Example 4. Nonempty $\mathcal{P}((\mathfrak{a}))$ has a few extreme points. Let $X$ be the interval $[-1, 1]$ and let $\mathfrak{a}$ consist of the even functions in $C(X)$. The equivalence classes are $\{0\}$ and the pairs $\{x, -x\}, 0 < x \leq 1$. From Theorem 1, the members of $\mathcal{P}((\mathfrak{a}))$ have the form $\mu_f(x) = s(|x|)f(|x|) + [1 - s(|x|)]f(-|x|), 0 < |x| \leq 1; \mu_f(0) = f(0)$, where $s$ is any function continuous on $0 < x \leq 1$ satisfying $0 \leq s(x) \leq 1, 0 < x \leq 1$ (e.g., $s(x) = \sin^2(1/x)$). It is easy to see that $\pi$ has exactly two cross sections; the corresponding extreme projections are $\mu_f(x) = f(|x|), -1 \leq x \leq 1$, and $\mu_f(x) = f(-|x|), -1 \leq x \leq 1$. The convex hull of these is obviously much smaller than $\mathcal{P}((\mathfrak{a}))$.

Example 5. Nonempty $\mathcal{P}((\mathfrak{a}))$ is the closed convex hull of its extreme points. In a number of applications [3], $X$ is the Stone space of a finite measure algebra, and in this event $\mathcal{P}((\mathfrak{a}))$ is nonempty and compact in a certain locally convex topology and hence (Krein-Milman) the closed convex hull of its extreme points. To be explicit let $(\Omega, \mathcal{F}, P)$ be a finite measure space, and let $L_\alpha(\Omega, \mathcal{F}, P)$ be the Banach algebra of $P$ essentially bounded scalar valued functions on $\Omega$. Then $L_\alpha(\Omega, \mathcal{F}, P)$ is isometrically algebraically isomorphic to $C(X)$, where the extremally disconnected compact Hausdorff space $X$ is the Stone space of the measure algebra $\mathcal{F}$ modulo $P$ null sets. Let $\mathcal{G}$ be a given sub-$\sigma$-field of $\mathcal{F}$. Under the isomorphism of $L_\alpha(\Omega, \mathcal{F}, P)$ and $C(X)$, the subalgebra $L_\alpha(\Omega, \mathcal{G}, P)$ of $\mathcal{G}$ measurable elements of $L_\alpha(\Omega, \mathcal{F}, P)$ is isomorphic to a closed self-adjoint subalgebra $\mathfrak{a}$ of $C(X)$ containing constants. A positive projection in $L_\alpha(\Omega, \mathcal{F}, P)$ onto $L_\alpha(\Omega, \mathcal{G}, P)$ (e.g., the conditional expectation operator with respect to $\mathcal{G}$) becomes a positive projection in $C(X)$ onto $\mathfrak{a}$. A proof that $\mathcal{P}((\mathfrak{a}))$ is compact in a suitable topology can be obtained as an easy modification of the proof of Theorem 8.4 of reference [3]; we omit the details.

Example 6. Nonempty noncompact $\mathcal{P}((\mathfrak{a}))$ is the closed convex hull of its extreme points. Let $X$ be the unit square $\{(x, y): 0 \leq x, y \leq 1\}$. Let $\mathfrak{a}$ consist of the functions of $x$ only. From Theorem 1, each $\mu \in \mathcal{P}((\mathfrak{a}))$ has the form $\mu_f(x, y) = \int_0^1 f(x, y')\sigma_x(dy'),$ where for each $0 \leq x \leq 1$, $\sigma_x$ is a regular Borel probability measure on the unit interval. The extreme projections are clearly $\mu_f(x, y) = f(x, y(x)), (x, y) \in X$, where $y(x)$ is any function continuous on and satisfying $0 \leq y(x) \leq 1$ on $0 \leq x \leq 1$. We will show that any given open weak operator neighborhood $N_\alpha$ of any given $\mu \in \mathcal{P}((\mathfrak{a}))$ contains a convex combination of extreme projections. Consider the projection $\bar{\sigma}$ whose corresponding $\bar{\sigma}$ is given by

$$\bar{\sigma}_x(y) = \frac{h}{1 + h} y + \frac{1}{h} \int_0^\nu \left[ \sigma_x(y' + y'h) - \sigma_x(y' + y'h - h) \right] dy';$$
we abbreviate $\sigma_x([0, y])$ to $\sigma_x(y)$. It is not hard to see that $\mu \in N_\mu$ if $\hbar > 0$ is sufficiently small. The measures $\sigma_x$ have densities satisfying

$$\frac{h}{1 + h} \leq \frac{\partial \sigma_x(y)}{\partial y} \leq \frac{h}{1 + h} + \frac{1}{h}, \quad 0 \leq x \leq 1.$$  

For each $0 < \alpha < 1$ and each $0 \leq x \leq 1$ let $y_\alpha(x)$ denote the (unique) solution of $\sigma_x(y_\alpha(x)) = \alpha$. The various properties of $\sigma$ insure that $y_\alpha(x)$ is continuous on $0 \leq x \leq 1$ for each fixed $0 < \alpha < 1$. Define $\mu_n \in \mathcal{P}(\alpha)$ by

$$\mu_n f(x, y) = \frac{1}{2^n - 1} \sum_{j=1}^{2^n-1} f(x, y_j, x), \quad (x, y) \in X.$$  

Riemann-Stieltjes integration theory shows that $\mu_n f(x, y)$ converges to $\mu f(x, y)$ for each $0 \leq x \leq 1$ and it follows that $\mu_n \in N_\mu$ for large enough $n$. Thus the convex hull of the extreme projections is dense in $\mathcal{P}(\alpha)$ in the weak operator topology. To see that $\mathcal{P}(\alpha)$ is not compact in this topology, consider the sequence $\mu_n$ defined by $\mu_n f(x, y) = f(x, y_n(x))$, where $y_n(x) = \min(nx, 1)$. This is a Cauchy sequence in the weak operator topology which does not converge to a member of $\mathcal{P}(\alpha)$; compactness fails because of incompleteness.

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**References**


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