ON EXTREME AVERAGING OPERATORS

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1. Introduction. With X a compact Hausdorff space, let $C(X)$ denote the Banach algebra of continuous scalar valued functions on X, with scalars either the real or the complex numbers, and with the usual pointwise operations and uniform norm. Let $\mathfrak{A}$ be a closed subalgebra of $C(X)$ containing constant functions, and complex conjugate functions in the complex case. We are concerned with the set $\mathcal{P}(\mathfrak{A})$ of all positive bounded projections in $C(X)$ onto fixed range space $\mathfrak{A}$.

It has been shown by Birkhoff [1] and others [2; 3] that the members of $\mathcal{P}(\mathfrak{A})$ are averaging operators, in the sense $\mu(f\mu g)=(\mu f)(\mu g)$, $f, g\in C(X)$, with structure of the following type. Subalgebras $\mathfrak{A}$ of the kind specified correspond 1-1 to upper semicontinuous decompositions of X into closed equivalence classes, the equivalence classes for a given $\mathfrak{A}$ consisting of the sets whose points are not separated by the functions in $\mathfrak{A}$. A given $\mu\in\mathcal{P}(\mathfrak{A})$ determines for each equivalence class a regular Borel probability measure living on the equivalence class and for each $f\in C(X)$ the value of μf on an equivalence class is obtained by averaging f over the equivalence class with respect to the associated measure.

As it turns out, $\mathcal{P}(\mathfrak{A})$ is a bounded convex set and we are interested in the extreme point properties of this convex set. Generalizing a result of Davis [4], we characterize the extreme points of $\mathcal{P}(\mathfrak{A})$ as those positive projections which are algebraic homomorphisms of $C(X)$ onto $\mathfrak{A}$. Examples are given where (i) $\mathcal{P}(\mathfrak{A})$ is empty (ii) nonempty $\mathcal{P}(\mathfrak{A})$ has no extreme points (iii) nonempty $\mathcal{P}(\mathfrak{A})$ has a few extreme points but is larger than their convex hull (iv) nonempty $\mathcal{P}(\mathfrak{A})$ is the closed convex hull of its extreme points. Situation (iv) obtains, in particular, when X is the extremally disconnected Stone space of a finite measure algebra.

2. Preliminaries. Let ϕ denote the equivalence in X associated with $\mathfrak{A}$. That is, $x_1\phi x_2$ if and only if $g(x_1)=g(x_2)$ for all $g\in\mathfrak{A}$. The quotient space $Y=X/\phi$ of equivalence classes is compact Hausdorff and the canonical continuous mapping $\pi: X\to Y$ is closed. The subalgebra $\mathfrak{A}$ is isometrically algebraically isomorphic to $C(Y)$ under the 1-1 correspondence $\rho: C(Y)\to\mathfrak{A}$ defined by

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$$\rho g(x) = g(\pi x), \qquad x \in X,\ g \in C(Y). \tag{1}$$

We represent the conjugate space $C^*(X)$ of bounded linear functionals on $C(X)$ as the space of regular bounded scalar valued (signed) Borel measures on X, with total variation norm. We denote by $\boldsymbol{X}$ the σ-field of Borel subsets of X. We denote by (f, λ) the value $(f, \lambda) = \int f(x)\lambda(dx)$ of $\lambda \in C^*(X)$ at $f \in C(X)$.

Proof of the following representation theory for members of $\mathcal{P}(\mathfrak{A})$ appears in [**1**; **2**; **3**]; we use the notation of Wright [**3**].

THEOREM 1. *The members of* $\mathcal{P}(\mathfrak{A})$ *correspond* 1-1 *to certain* $C^*(X)$ *valued functions on* Y, *as follows. If* $\sigma: Y \to C^*(X)$ *corresponds to* $\mu \in \mathcal{P}(\mathfrak{A})$ *then* σ *and* μ *are related by* (i), *and* σ *has properties* (ii)–(iv):

(2)
(i) $\mu f(x) = (f, \sigma_{\pi x}) = \int f(x')\sigma_{\pi x}(dx')$, $x \in X$, $f \in C(X)$,

(ii) $\sigma: Y \to C^*(X)$ *is continuous with the* $C(X)$ (*weak**) *topology in* $C^*(X)$,

(iii) $\sigma_y \geqq 0$, $\sigma_y(X) = 1$, $y \in Y$,

(iv) $\sigma_y(E) = \sigma_y(E \cap \pi^{-1}y)$, $E \in \boldsymbol{X}$, $y \in Y$.

3. **Extreme points.** A cross section of the mapping $\pi: X \to Y$ is any continuous function $\gamma: Y \to X$ such that $\pi\gamma$ is the identity mapping on Y. When the measure σ_y of (2) is concentrated at a single point $\gamma(y) \in \pi^{-1}y$ for each $y \in Y$ then $\gamma: Y \to X$ is a cross section of π, from (2.ii) and (2.iii). The corresponding projection is an algebraic homomorphism of $C(X)$ onto $\mathfrak{A}$; in fact, we have (as noted by Wright [**3**])

THEOREM 2. *The members of* $\mathcal{P}(\mathfrak{A})$ *which are homomorphisms of* $C(X)$ *onto* $\mathfrak{A}$ *correspond* 1-1 *to cross sections of* $\pi: X \to Y$.

The connection with extreme points is given by

THEOREM 3. *A positive projection* $\mu \in \mathcal{P}(\mathfrak{A})$ *is an extreme point of* $\mathcal{P}(\mathfrak{A})$ *if and only if* μ *is an algebraic homomorphism of* $C(X)$ *onto* $\mathfrak{A}$.

PROOF. The if part of the statement is due to Wright [**3**]; we reproduce the proof for the sake of completeness. Suppose $\mu \in \mathcal{P}(\mathfrak{A})$ is a homomorphism. In representation (2), this becomes $(fg, \sigma_y) = (f, \sigma_y)(g, \sigma_y)$, $y \in Y$, $f, g \in C(X)$, which is to say that for each $y \in Y$ the functional $\sigma_y \in C^*(X)$ is a homomorphism of $C(X)$ into the scalars, and onto because (2.iii) implies $(1, \sigma_y) = 1$. It is well known that σ_y must then be for each y a unit point measure, that is, an extreme point of the set of regular Borel probability measures on X. It follows at once that μ is necessarily an extreme point of $\mathcal{P}(\mathfrak{A})$. Conversely, suppose μ is an extreme point of $\mathcal{P}(\mathfrak{A})$. We will show that

the corresponding σ has the property that for each $y \in Y$ and $E \in \boldsymbol{X}$, either $\sigma_y(E)=0$ or $\sigma_y(E)=1$. It will then follow from well-known arguments that σ_y is for each y a homomorphism of $C(X)$ onto the scalars, and hence μ a homomorphism onto $\mathfrak{A}$. Suppose to the contrary that $0<\sigma_{y_0}(E_0)<1$ obtains for some $y_0 \in Y$, $E_0 \in \boldsymbol{X}$. Since σ_{y_0} is a regular Borel probability measure, there exist closed sets $F_1 \subset E_0$, $F_2 \subset E_0'$, such that $\sigma_{y_0}(F_1)>0$, $\sigma_{y_0}(F_2)>0$. Because X is normal, there exists $f_1 \in C(X)$ such that $f_1(F_1)=1$, $f_1(F_2)=0$, $0 \leqq f_1 \leqq 1$. With $f_2=1-f_1$, define $\tau^{(i)}$: $Y \to C^*(X)$ by

$$\tau_y^{(i)}(E) = \int_E f_i(x)\sigma_y(dx), \qquad y \in Y,\ E \in \boldsymbol{X},\ i = 1,\ 2,$$

noting that $\tau^{(1)}+\tau^{(2)}=\sigma$. The measure $\tau_y^{(i)}$ lives on $\pi^{-1}y$, clearly, and from $(f, \tau_y^{(i)})=(ff_i, \sigma_y)$, $y \in Y$, $f \in C(X)$ (Radon-Nikodym) it follows that each $\tau^{(i)}$ induces a positive bounded linear mapping of $C(X)$ into $C(Y)$.

In particular, the nonnegative real functions $\tau_y^{(i)}(X)=(1, \tau_y^{(i)})=(f_i, \sigma_y)$ are continuous on Y, and moreover, $\tau_{y_0}^{(i)}(X) \geqq \sigma_{y_0}(F_i)>0$, $i=1, 2$. For an arbitrary $a>0$ there exists then an open neighborhood V of y_0 such that

$$\left| \tau_y^{(i)}(X) - \tau_{y_0}^{(i)}(X) \right| < a\tau_{y_0}^{(i)}(X), \qquad i = 1, 2,\ y \in V.$$

Because Y is completely regular, there exists $h \in C(Y)$ such that $h(y_0)=1$, $h(V')=0$, $0 \leqq h \leqq 1$. With b to be chosen presently, define

$$\sigma_y^{(i)}(E) = \left[\frac{bh(y)}{\tau_{y_0}^{(i)}(X)}\right]\tau_y^{(i)}(E) + \left[1 - \frac{bh(y)}{\tau_{y_0}^{(i)}(X)}\,\tau_y^{(i)}(X)\right]\sigma_y(E),$$
$$y \in Y,\ E \in \boldsymbol{X},\ i = 1, 2.$$

Now with $0<b<(a+1)^{-1}$, the continuous coefficients in square brackets above are nonnegative real on Y, and it is readily verified that each $\sigma^{(i)}$ induces a positive projection in $C(X)$ onto $\mathfrak{A}$. A straightforward calculation gives

$$(f_1, \sigma_{y_0}^{(1)}) - (f_1, \sigma_{y_0}^{(2)}) = b\,\frac{(f_1^2, \sigma_{y_0}) - (f_1, \sigma_{y_0})^2}{(f_1, \sigma_{y_0})(f_2, \sigma_{y_0})},$$

and since $f_1 \neq$const almost everywhere $[\sigma_{y_0}]$, the above difference does not vanish; that is, $\sigma^{(1)}$ and $\sigma^{(2)}$ correspond to distinct means. Finally, the identity $\sigma=\tau_{y_0}^{(1)}(X)\sigma^{(1)}+\tau_{y_0}^{(2)}(X)\sigma^{(2)}$ shows that σ does not correspond to an extreme mean, contrary to hypothesis. Hence it must be the case that σ takes values 0 and 1 only, implying that

each σ_y is a homomorphism. (Proof as in [5, p. 444]; we omit the details.)

COROLLARY. *Extreme points of* $\mathcal{P}(\mathfrak{A})$ *correspond* 1-1 *to cross sections of* $\pi: X \to Y$.

4. Some examples. The author is indebted to E. A. Michael for Example 1 and for a suggestion which led to Examples 2 and 3.

EXAMPLE 1. $\mathcal{P}(\mathfrak{A})$ is empty. Let X be the interval $[0, 3]$ and let $\mathfrak{A}$ consist of the functions in $C(X)$ which are constant on the subinterval $[1, 2]$. The equivalence classes consist of the interval $[1, 2]$ and the remaining individual points of X. From Theorem 1, $\mu \in \mathcal{P}(\mathfrak{A})$ would have the form

$$\mu f(x) = \int_1^2 f(x')\sigma(dx'),\ x \in [1, 2];\ \mu f(x) = f(x),\ x \in X - [1, 2].$$

For no measure σ is μf continuous if $f(1) \neq f(2)$, and it follows that $\mathcal{P}(\mathfrak{A})$ is empty.

EXAMPLE 2. Nonempty $\mathcal{P}(\mathfrak{A})$ has no extreme points. Let X be the interval $[0, 2]$ with endpoints identified, i.e., a circle, and let $\mathfrak{A}$ consist of the functions g in $C(X)$ satisfying $g(x) = g(x+1)$, $0 \leqq x < 1$. The equivalence classes are the pairs $\{x, x+1\}$, $0 \leqq x < 1$, and Y may be represented as the interval $[0, 1]$ with endpoints identified. Let $s(x)$, $0 \leqq x \leqq 2$, be any continuous function satisfying $0 \leqq s(x) \leqq 1$, $0 \leqq x \leqq 2$, and $s(x) + s(x+1) = 1$, $0 \leqq x \leqq 1$. Then $\mu: C(X) \to \mathfrak{A}$ defined by $\mu f(x) = \mu f(x+1) = s(x)f(x) + s(x+1)f(x+1)$, $0 \leqq x \leqq 1$, is a positive bounded projection onto $\mathfrak{A}$, as is easily verified. It is not hard to see that the canonical projection $\pi: X \to Y$ in this example has no cross sections. It follows from Theorem 3 corollary that $\mathcal{P}(\mathfrak{A})$ has no extreme points.

The mapping π is open in Example 2, where projections exist, and not open in Example 1, where projections do not exist, and Michael [6] has shown that sufficient conditions for $\mathcal{P}(\mathfrak{A})$ to be nonempty are that π be open and X be metric. Openness of π is not necessary for the existence of projections, however, as the next example shows.

EXAMPLE 3. Nonempty $\mathcal{P}(\mathfrak{A})$ has no extreme points; π is not open. Let X be the interval $[0, 2]$, but now without identifications. Let $\mathfrak{A}$ consist of the functions g in $C(X)$ satisfying $g(x) = g(x+1)$, $0 \leqq x \leqq 1$. The equivalence classes are the pairs $\{x, x+1\}$, $0 < x < 1$, and the triple $\{0, 1, 2\}$. Again, Y is a circle; π is no longer open. Projections onto $\mathfrak{A}$ exist and have the same form as in Example 2, except that s must now also satisfy $s(0) = s(2) = 0$. It is still the case that π has no cross sections, whence $\mathcal{P}(\mathfrak{A})$ has no extreme points.

EXAMPLE 4. Nonempty $\mathcal{P}(\mathcal{A})$ has a few extreme points. Let X be the interval $[-1, 1]$ and let $\mathcal{A}$ consist of the even functions in $C(X)$. The equivalence classes are $\{0\}$ and the pairs $\{x, -x\}$, $0<x\leqq 1$. From Theorem 1, the members of $\mathcal{P}(\mathcal{A})$ have the form $\mu f(x) = s(|x|)f(|x|)+[1-s(|x|)]f(-|x|)$, $0<|x|\leqq 1$; $\mu f(0)=f(0)$, where s is any function continuous on $0<x\leqq 1$ satisfying $0\leqq s(x)\leqq 1$, $0<x\leqq 1$ (e.g., $s(x)=\sin^2(1/x)$). It is easy to see that π has exactly two cross sections; the corresponding extreme projections are $\mu f(x) = f(|x|)$, $-1\leqq x\leqq 1$, and $\mu f(x)=f(-|x|)$, $-1\leqq x\leqq 1$. The convex hull of these is obviously much smaller than $\mathcal{P}(\mathcal{A})$.

EXAMPLE 5. Nonempty $\mathcal{P}(\mathcal{A})$ is the closed convex hull of its extreme points. In a number of applications [3], X is the Stone space of a finite measure algebra, and in this event $\mathcal{P}(\mathcal{A})$ is nonempty and compact in a certain locally convex topology and hence (Krein-Milman) the closed convex hull of its extreme points. To be explicit let $(\Omega, \mathfrak{F}, P)$ be a finite measure space, and let $L_\infty(\Omega, \mathfrak{F}, P)$ be the Banach algebra of P essentially bounded scalar valued functions on Ω. Then $L_\infty(\Omega, \mathfrak{F}, P)$ is isometrically algebraically isomorphic to $C(X)$, where the extremally disconnected compact Hausdorff space X is the Stone space of the measure algebra $\mathfrak{F}$ modulo P null sets. Let $\mathcal{G}$ be a given sub-σ-field of $\mathfrak{F}$. Under the isomorphism of $L_\infty(\Omega, \mathfrak{F}, P)$ and $C(X)$, the subalgebra $L_\infty(\Omega, \mathcal{G}, P)$ of $\mathcal{G}$ measurable elements of $L_\infty(\Omega, \mathfrak{F}, P)$ is isomorphic to a closed self-adjoint subalgebra $\mathcal{A}$ of $C(X)$ containing constants. A positive projection in $L_\infty(\Omega, \mathfrak{F}, P)$ onto $L_\infty(\Omega, \mathcal{G}, P)$ (e.g., the conditional expectation operator with respect to $\mathcal{G}$) becomes a positive projection in $C(X)$ onto $\mathcal{A}$. A proof that $\mathcal{P}(\mathcal{A})$ is compact in a suitable topology can be obtained as an easy modification of the proof of Theorem 8.4 of reference [3]; we omit the details.

EXAMPLE 6. Nonempty noncompact $\mathcal{P}(\mathcal{A})$ is the closed convex hull of its extreme points. Let X be the unit square $\{(x, y): 0\leqq x, y\leqq 1\}$. Let $\mathcal{A}$ consist of the functions of x only. From Theorem 1, each $\mu\in\mathcal{P}(\mathcal{A})$ has the form $\mu f(x, y)=\int_0^1 f(x, y')\sigma_x(dy')$, where for each $0\leqq x\leqq 1$, σ_x is a regular Borel probability measure on the unit interval. The extreme projections are clearly $\mu f(x, y)=f(x, y(x))$, $(x, y)\in X$, where $y(x)$ is any function continuous on and satisfying $0\leqq y(x)\leqq 1$ on $0\leqq x\leqq 1$. We will show that any given open weak operator neighborhood N_μ of any given $\mu\in\mathcal{P}(\mathcal{A})$ contains a convex combination of extreme projections. Consider the projection $\bar{\mu}$ whose corresponding $\bar{\sigma}$ is given by

$$\bar{\sigma}_x(y) = \frac{h}{1+h}y + \frac{1}{h}\int_0^y [\sigma_x(y'+y'h) - \sigma_x(y'+y'h-h)]dy';$$

we abbreviate $\sigma_x([0, y])$ to $\sigma_x(y)$. It is not hard to see that $\bar{\mu}\in N_\mu$ if $h>0$ is sufficiently small. The measures $\bar{\sigma}_x$ have densities satisfying

$$\frac{h}{1+h} \leqq \frac{\partial\bar{\sigma}_x(y)}{\partial y} \leqq \frac{h}{1+h} + \frac{1}{h}, \qquad 0 \leqq x \leqq 1.$$

For each $0<\alpha<1$ and each $0\leqq x\leqq 1$ let $y_\alpha(x)$ denote the (unique) solution of $\bar{\sigma}_x(y_\alpha(x))=\alpha$. The various properties of $\bar{\sigma}$ insure that $y_\alpha(x)$ is continuous on $0\leqq x\leqq 1$ for each fixed $0<\alpha<1$. Define $\mu_n\in\mathcal{P}(\mathfrak{A})$ by

$$\mu_n f(x, y) = \frac{1}{2^n - 1}\sum_{j=1}^{2^n-1} f(x, y_{j/2^n}(x)), \qquad (x, y) \in X.$$

Riemann-Stieltjes integration theory shows that $\mu_n f(x, y)$ converges to $\bar{\mu}f(x, y)$ for each $0\leqq x\leqq 1$ and it follows that $\mu_n\in N_\mu$ for large enough n. Thus the convex hull of the extreme projections is dense in $\mathcal{P}(\mathfrak{A})$ in the weak operator topology. To see that $\mathcal{P}(\mathfrak{A})$ is not compact in this topology, consider the sequence μ_n defined by $\mu_n f(x, y) = f(x, y_n(x))$, where $y_n(x)=\min(nx, 1)$. This is a Cauchy sequence in the weak operator topology which does not converge to a member of $\mathcal{P}(\mathfrak{A})$; compactness fails because of incompleteness.

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