ON EXTREME AVERAGING OPERATORS

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1. Introduction. With \( X \) a compact Hausdorff space, let \( C(X) \) denote the Banach algebra of continuous scalar valued functions on \( X \), with scalars either the real or the complex numbers, and with the usual pointwise operations and uniform norm. Let \( \mathcal{A} \) be a closed subalgebra of \( C(X) \) containing constant functions, and complex conjugate functions in the complex case. We are concerned with the set \( \mathcal{P}(\mathcal{A}) \) of all positive bounded projections in \( C(X) \) onto fixed range space \( \mathcal{A} \).

It has been shown by Birkhoff [1] and others [2; 3] that the members of \( \mathcal{P}(\mathcal{A}) \) are averaging operators, in the sense \( \mu(fg) = (\mu f)(\mu g) \), \( f, g \in C(X) \), with structure of the following type. Subalgebras \( \mathcal{A} \) of the kind specified correspond 1-1 to upper semicontinuous decompositions of \( X \) into closed equivalence classes, the equivalence classes for a given \( \mathcal{A} \) consisting of the sets whose points are not separated by the functions in \( \mathcal{A} \). A given \( \mu \in \mathcal{P}(\mathcal{A}) \) determines for each equivalence class a regular Borel probability measure living on the equivalence class and for each \( f \in C(X) \) the value of \( \mu f \) on an equivalence class is obtained by averaging \( f \) over the equivalence class with respect to the associated measure.

As it turns out, \( \mathcal{P}(\mathcal{A}) \) is a bounded convex set and we are interested in the extreme point properties of this convex set. Generalizing a result of Davis [4], we characterize the extreme points of \( \mathcal{P}(\mathcal{A}) \) as those positive projections which are algebraic homomorphisms of \( C(X) \) onto \( \mathcal{A} \). Examples are given where (i) \( \mathcal{P}(\mathcal{A}) \) is empty (ii) nonempty \( \mathcal{P}(\mathcal{A}) \) has no extreme points (iii) nonempty \( \mathcal{P}(\mathcal{A}) \) has a few extreme points but is larger than their convex hull (iv) nonempty \( \mathcal{P}(\mathcal{A}) \) is the closed convex hull of its extreme points. Situation (iv) obtains, in particular, when \( X \) is the extremally disconnected Stone space of a finite measure algebra.

2. Preliminaries. Let \( \phi \) denote the equivalence in \( X \) associated with \( \mathcal{A} \). That is, \( x_1 \phi x_2 \) if and only if \( g(x_1) = g(x_2) \) for all \( g \in \mathcal{A} \). The quotient space \( Y = X/\phi \) of equivalence classes is compact Hausdorff and the canonical continuous mapping \( \pi: X \to Y \) is closed. The subalgebra \( \mathcal{A} \) is isometrically algebraically isomorphic to \( C(Y) \) under the 1-1 correspondence \( \rho: C(Y) \to \mathcal{A} \) defined by

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We represent the conjugate space $C^*(X)$ of bounded linear functionals on $C(X)$ as the space of regular bounded scalar valued (signed) Borel measures on $X$, with total variation norm. We denote by $X$ the $\sigma$-field of Borel subsets of $X$. We denote by $(f, \lambda)$ the value $(f, \lambda) = \int f(x)\lambda(dx)$ of $\lambda \in C^*(X)$ at $f \in C(X)$.

Proof of the following representation theory for members of $\mathcal{P}(\alpha)$ appears in [1; 2; 3]; we use the notation of Wright [3].

THEOREM 1. The members of $\mathcal{P}(\alpha)$ correspond 1-1 to certain $C^*(X)$ valued functions on $Y$, as follows. If $\sigma: Y \to C^*(X)$ corresponds to $\mu \in \mathcal{P}(\alpha)$ then $\sigma$ and $\mu$ are related by (i), and $\sigma$ has properties (ii)-(iv):

(i) $\mu f(x) = (f, \sigma_x) = \int f(x')\sigma_{x'}(dx'), \ x \in X, f \in C(X)$,

(ii) $\sigma: Y \to C^*(X)$ is continuous with the $C(X)$ (weak*) topology in $C^*(X)$,

(iii) $\sigma_y \geq 0, \sigma_y(X) = 1, y \in Y$,

(iv) $\sigma_y(E) = \sigma_y(E \cap \pi^{-1}y), E \subseteq X, y \in Y$.

3. Extreme points. A cross section of the mapping $\pi: X \to Y$ is any continuous function $\gamma: Y \to X$ such that $\pi \gamma$ is the identity mapping on $Y$. When the measure $\sigma_y$ of (2) is concentrated at a single point $\gamma(y) \subseteq \pi^{-1}y$ for each $y \in Y$ then $\gamma: Y \to X$ is a cross section of $\pi$, from (2.ii) and (2.iii). The corresponding projection is an algebraic homomorphism of $C(X)$ onto $\alpha$; in fact, we have (as noted by Wright [3])

THEOREM 2. The members of $\mathcal{P}(\alpha)$ which are homomorphisms of $C(X)$ onto $\alpha$ correspond 1-1 to cross sections of $\pi: X \to Y$.

The connection with extreme points is given by

THEOREM 3. A positive projection $\mu \in \mathcal{P}(\alpha)$ is an extreme point of $\mathcal{P}(\alpha)$ if and only if $\mu$ is an algebraic homomorphism of $C(X)$ onto $\alpha$.

Proof. The if part of the statement is due to Wright [3]; we reproduce the proof for the sake of completeness. Suppose $\mu \in \mathcal{P}(\alpha)$ is a homomorphism. In representation (2), this becomes $(fg, \sigma_y) = (f, \sigma_y)(g, \sigma_y), y \in Y, f, g \in C(X)$, which is to say that for each $y \in Y$ the functional $\sigma_y \in C^*(X)$ is a homomorphism of $C(X)$ into the scalars, and onto because (2.iii) implies $(1, \sigma_y) = 1$. It is well known that $\sigma_y$ must then be for each $y$ a unit point measure, that is, an extreme point of the set of regular Borel probability measures on $X$. It follows at once that $\mu$ is necessarily an extreme point of $\mathcal{P}(\alpha)$. Conversely, suppose $\mu$ is an extreme point of $\mathcal{P}(\alpha)$. We will show that
the corresponding $\sigma$ has the property that for each $y \in Y$ and $E \subseteq X$, either $\sigma_y(E) = 0$ or $\sigma_y(E) = 1$. It will then follow from well-known arguments that $\sigma_y$ is for each $y$ a homomorphism of $C(X)$ onto the scalars, and hence $\mu$ a homomorphism onto $G$. Suppose to the contrary that $0 < \sigma_y(E_0) < 1$ obtains for some $y_0 \in Y$, $E_0 \subseteq X$. Since $\sigma_y$ is a regular Borel probability measure, there exist closed sets $F_1 \subseteq E_0$, $F_2 \subseteq E_0'$ such that $\sigma_y(F_1) > 0$, $\sigma_y(F_2) > 0$. Because $X$ is normal, there exists $f_1 \in C(X)$ such that $f_1(F_1) = 1$, $f_1(F_2) = 0$, $0 \leq f_1 \leq 1$. With $f_1 = 1 - f_1$, define $\tau^{(i)} : Y \to C(Y)$ by

$$
\tau^{(i)}(y)(E) = \int_E f_i(x) \sigma_y(dx), \quad y \in Y, \ E \subseteq X, \ i = 1, 2,
$$

noting that $\tau^{(1)} + \tau^{(2)} = \sigma$. The measure $\tau^{(i)}_y$ lives on $\pi^{-1}y$, clearly, and from $(f_i, \tau^{(i)}_y) = (f, \sigma_y)$, $y \in Y$, $f \in C(X)$ (Radon-Nikodym) it follows that each $\tau^{(i)}$ induces a positive bounded linear mapping of $C(X)$ into $C(Y)$.

In particular, the nonnegative real functions $\tau^{(i)}_y(X) = 1$, $\tau^{(0)}_y(X) = (f_i, \sigma_y)$ are continuous on $Y$, and moreover, $\tau^{(i)}_y(X) \geq \sigma_y(F_i) > 0$, $i = 1, 2$. For an arbitrary $a > 0$ there exists then an open neighborhood $V$ of $y_0$ such that

$$
| \tau^{(i)}(X) - \tau^{(i)}(X) | < a \tau^{(i)}(X), \quad i = 1, 2, \ y \in V.
$$

Because $Y$ is completely regular, there exists $h \in C(Y)$ such that $h(y_0) = 1$, $h(V') = 0$, $0 \leq h \leq 1$. With $b$ to be chosen presently, define

$$
\sigma^{(i)}_y(E) = \left[ \frac{bh(y)}{\tau^{(0)}_y(X)} \right] \tau^{(i)}_y(E) + \left[ 1 - \frac{bh(y)}{\tau^{(0)}_y(X)} \right] \sigma_y(E), \quad y \in Y, \ E \subseteq X, \ i = 1, 2.
$$

Now with $0 < b < (a + 1)^{-1}$, the continuous coefficients in square brackets above are nonnegative real on $Y$, and it is readily verified that each $\sigma^{(i)}$ induces a positive projection in $C(X)$ onto $a$. A straightforward calculation gives

$$
(f_1, \sigma^{(1)}_y) - (f_1, \sigma^{(2)}_y) = b \frac{(f_1, \sigma_y) - (f_1, \sigma_y)^2}{(f_1, \sigma_y)(f_2, \sigma_y)},
$$

and since $f_1 \neq \text{const}$ almost everywhere $[\sigma_y]$, the above difference does not vanish; that is, $\sigma^{(1)}$ and $\sigma^{(2)}$ correspond to distinct means. Finally, the identity $\sigma = \tau^{(0)}_y(X) \tau^{(1)} + \tau^{(2)}_y(X) \sigma^{(0)}$ shows that $\sigma$ does not correspond to an extreme mean, contrary to hypothesis. Hence it must be the case that $\sigma$ takes values 0 and 1 only, implying that
each $\sigma_x$ is a homomorphism. (Proof as in [5, p. 444]; we omit the details.)

**Corollary.** Extreme points of $\vartheta(\alpha)$ correspond 1-1 to cross sections of $\pi: X \to Y$.

### 4. Some examples

The author is indebted to E. A. Michael for Example 1 and for a suggestion which led to Examples 2 and 3.

**Example 1.** $\vartheta(\alpha)$ is empty. Let $X$ be the interval $[0, 3]$ and let $\alpha$ consist of the functions in $C(X)$ which are constant on the subinterval $[1, 2]$. The equivalence classes consist of the interval $[1, 2]$ and the remaining individual points of $X$. From Theorem 1, $\mu \in \vartheta(\alpha)$ would have the form

$$\mu f(x) = \int_1^2 f(x')\sigma(dx'), \ x \in [1, 2]; \mu f(x) = f(x), \ x \in X - [1, 2].$$

For no measure $\sigma$ is $\mu f$ continuous if $f(1) \neq f(2)$, and it follows that $\vartheta(\alpha)$ is empty.

**Example 2.** Nonempty $\vartheta(\alpha)$ has no extreme points. Let $X$ be the interval $[0, 2]$ with endpoints identified, i.e., a circle, and let $\alpha$ consist of the functions $g$ in $C(X)$ satisfying $g(x) = g(x + 1), 0 \leq x < 1$. The equivalence classes are the pairs $\{x, x + 1\}, 0 \leq x < 1$, and $Y$ may be represented as the interval $[0, 1]$ with endpoints identified. Let $s(x), 0 \leq s(x) \leq 2, be any continuous function satisfying $0 \leq s(x) \leq 1, 0 \leq x \leq 2, and s(x) + s(x + 1) = 1, 0 \leq x \leq 1$. Then $\mu: C(X) \to \alpha$ defined by $\mu f(x) = \mu f(x + 1) = s(x)f(x) + s(x + 1)f(x + 1), 0 \leq x \leq 1$, is a positive bounded projection onto $\alpha$, as is easily verified. It is not hard to see that the canonical projection $\pi: X \to Y$ in this example has no cross sections. It follows from Theorem 3 corollary that $\vartheta(\alpha)$ has no extreme points.

The mapping $\pi$ is open in Example 2, where projections exist, and not open in Example 1, where projections do not exist, and Michael [6] has shown that sufficient conditions for $\vartheta(\alpha)$ to be nonempty are that $\pi$ be open and $X$ be metric. Openness of $\pi$ is not necessary for the existence of projections, however, as the next example shows.

**Example 3.** Nonempty $\vartheta(\alpha)$ has no extreme points; $\pi$ is not open. Let $X$ be the interval $[0, 2]$, but now without identifications. Let $\alpha$ consist of the functions $g$ in $C(X)$ satisfying $g(x) = g(x + 1), 0 \leq x \leq 1$. The equivalence classes are the pairs $\{x, x + 1\}, 0 < x < 1$, and the triple $\{0, 1, 2\}$. Again, $Y$ is a circle; $\pi$ is no longer open. Projections onto $\alpha$ exist and have the same form as in Example 2, except that $s$ must now also satisfy $s(0) = s(2) = 0$. It is still the case that $\pi$ has no cross sections, whence $\vartheta(\alpha)$ has no extreme points.
Example 4. Nonempty \( \varphi(\mathcal{A}) \) has a few extreme points. Let \( X \) be the interval \([-1, 1]\) and let \( \mathcal{A} \) consist of the even functions in \( C(X) \). The equivalence classes are \( \{0\} \) and the pairs \( \{x, -x\} \), \( 0 < x \leq 1 \).

From Theorem 1, the members of \( \varphi(\mathcal{A}) \) have the form \( \mu f(x) = s(|x|) f(|x|) + [1 - s(|x|)] f(-|x|), 0 < |x| \leq 1; \mu f(0) = f(0) \), where \( s \) is any function continuous on \( 0 < x \leq 1 \) satisfying \( 0 \leq s(x) \leq 1 \), \( 0 < x \leq 1 \) (e.g., \( s(x) = \sin^2(1/x) \)). It is easy to see that \( \sigma \) has exactly two cross sections; the corresponding extreme projections are \( \mu f(x) = f(|x|), -1 \leq x \leq 1 \), and \( \mu f(x) = f(-|x|), -1 \leq x \leq 1 \). The convex hull of these is obviously much smaller than \( \varphi(\mathcal{A}) \).

Example 5. Nonempty \( \varphi(\mathcal{A}) \) is the closed convex hull of its extreme points. In a number of applications \([3]\), \( X \) is the Stone space of a finite measure algebra, and in this event \( \varphi(\mathcal{A}) \) is nonempty and compact in a certain locally convex topology and hence (Krein-Milman) the closed convex hull of its extreme points. To be explicit let \( (\Omega, \mathcal{F}, P) \) be a finite measure space, and let \( L_{\infty}(\Omega, \mathcal{F}, P) \) be the Banach algebra of \( P \) essentially bounded scalar valued functions on \( \Omega \). Then \( L_{\infty}(\Omega, \mathcal{F}, P) \) is isometrically algebraically isomorphic to \( C(X) \), where the extremally disconnected compact Hausdorff space \( X \) is the Stone space of the measure algebra \( \mathcal{F} \) modulo \( P \) null sets. Let \( \mathcal{G} \) be a given sub-\( \sigma \)-field of \( \mathcal{F} \). Under the isomorphism of \( L_{\infty}(\Omega, \mathcal{F}, P) \) and \( C(X) \), the subalgebra \( L_{\infty}(\Omega, \mathcal{G}, P) \) of \( \mathcal{G} \) measurable elements of \( L_{\infty}(\Omega, \mathcal{F}, P) \) is isomorphic to a closed self-adjoint subalgebra \( \mathcal{A} \) of \( C(X) \) containing constants. A positive projection in \( L_{\infty}(\Omega, \mathcal{F}, P) \) onto \( L_{\infty}(\Omega, \mathcal{G}, P) \) (e.g., the conditional expectation operator with respect to \( \mathcal{G} \)) becomes a positive projection in \( C(X) \) onto \( \mathcal{A} \). A proof that \( \varphi(\mathcal{A}) \) is compact in a suitable topology can be obtained as an easy modification of the proof of Theorem 8.4 of reference \([3]\); we omit the details.

Example 6. Nonempty noncompact \( \varphi(\mathcal{A}) \) is the closed convex hull of its extreme points. Let \( X \) be the unit square \( \{(x, y): 0 \leq x, y \leq 1\} \). Let \( \mathcal{A} \) consist of the functions of \( x \) only. From Theorem 1, each \( \mu \in \varphi(\mathcal{A}) \) has the form \( \mu f(x, y) = \int_0^y f(x, y') s_\gamma(dy') \), where for each \( 0 \leq y \leq 1 \), \( s_\gamma \) is a regular Borel probability measure on the unit interval. The extreme projections are clearly \( \mu f(x, y) = f(x, y(x)), (x, y) \in X \), where \( y(x) \) is any function continuous on and satisfying \( 0 \leq y(x) \leq 1 \) on \( 0 \leq x \leq 1 \). We will show that any given open weak operator neighborhood \( N_\varepsilon \) of any given \( \mu \in \varphi(\mathcal{A}) \) contains a convex combination of extreme projections. Consider the projection \( \mu \) whose corresponding \( \tilde{s} \) is given by

\[
\tilde{s}_s(y) = \frac{h}{1 + h} y + \frac{1}{h} \int_0^y [s_\gamma(y' + y'h) - s_\gamma(y' + y'h - h)] dy';
\]
we abbreviate $\sigma_x([0, y])$ to $\sigma_x(y)$. It is not hard to see that $\mu \in N_\mu$ if $h > 0$ is sufficiently small. The measures $\sigma_x$ have densities satisfying

$$\frac{h}{1 + h} \leq \frac{\partial \sigma_x(y)}{\partial y} \leq \frac{h}{1 + h} + \frac{1}{h}, \quad 0 \leq x \leq 1.$$

For each $0 < \alpha < 1$ and each $0 \leq x \leq 1$ let $y_\alpha(x)$ denote the (unique) solution of $\sigma_x(y_\alpha(x)) = \alpha$. The various properties of $\sigma$ insure that $y_\alpha(x)$ is continuous on $0 \leq x \leq 1$ for each fixed $0 < \alpha < 1$. Define $\mu_n \in \Phi(\alpha)$ by

$$\mu_n f(x, y) = \frac{1}{2^n - 1} \sum_{j=1}^{2^n-1} f(x, y_{j/2^n}(x)), \quad (x, y) \in X.$$

Riemann-Stieltjes integration theory shows that $\mu_n f(x, y)$ converges to $\mu f(x, y)$ for each $0 \leq x \leq 1$ and it follows that $\mu_n \in N_\mu$ for large enough $n$. Thus the convex hull of the extreme projections is dense in $\Phi(\alpha)$ in the weak operator topology. To see that $\Phi(\alpha)$ is not compact in this topology, consider the sequence $\mu_n$ defined by $\mu_n f(x, y) = f(x, y_n(x))$, where $y_n(x) = \min(nx, 1)$. This is a Cauchy sequence in the weak operator topology which does not converge to a member of $\Phi(\alpha)$; compactness fails because of incompleteness.

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**References**