ISOTOPY AND PARASTROPHY OF QUASIGROUPS

R. ARTZY

1. It has been noted that every quasigroup $(Q, \cdot)$ belongs to a set of 6 quasigroups, called adjugate by Fisher and Yates [4], conjugate by Stein [6], *parastrophic* by Sade [5]. If in $(Q, \cdot)$, $xy = z$, then in the parastrophic quasigroups $(x\pi)(\pi)(y\pi) = z\pi$, where π is one of the 6 permutations of $\{x, y, z\}$, $v\pi$ the image of $v \in \{x, y, z\}$ under π, and (π) the operation in the parastrophic quasigroup (Q, π). Let ρ be the permutation $\left(\begin{smallmatrix} xyz \\ yzx \end{smallmatrix}\right)$, $\tau = \left(\begin{smallmatrix} xyz \\ yxz \end{smallmatrix}\right)$. Then every π is generated by ρ and τ, the generators of the symmetric group S_3, and defining relations are $\tau^2 = (\rho\tau)^2 = \rho^3 = I$, the identity permutation. The mappings of quasigroups (Q, π) on each other will be called *parastrophisms*.

If α, β, γ are permutations of the elements of Q, then (Q, κ) with the operation (κ) such that $(x\alpha)(\kappa)(y\beta) = (xy)\gamma$ is an isotope of $(Q, \cdot)$. The mappings of the quasigroups (Q, κ) onto each other are the isotopisms of Q, with the notation $\kappa = [\alpha, \beta, \gamma]$ for the isotopism $(Q, \cdot) \to (Q, \kappa)$. Since the parastrophisms and the isotopisms are permutations, both the parastrophisms and the isotopisms of Q form groups, the parastrophism group being isomorphic to S_3 or to one of its subgroups. The parastrophism $(Q, \cdot) \to (Q, \pi)$, induced by the permutation π, will also be called π. The group generated by all the parastrophisms and isotopisms of Q will be called G.

THEOREM 1. *The isotopism group, T, of Q is normal in G.*

PROOF. Since G is generated by ρ, τ and T, it is sufficient to prove $\rho^{-1}T\rho \in T$, $\rho T\rho^{-1} \in T$ and, in view of $\tau = \tau^{-1}$, $\tau T\tau \in T$. Let $[\alpha, \beta, \gamma]$ be an isotopism. Then, by a well-known rule for permutations, $\rho^{-1}[\alpha, \beta, \gamma]\rho = [\beta, \gamma, \alpha] \in T$ and $\rho[\alpha, \beta, \gamma]\rho^{-1} = (\rho^{-1})^{-1}[\alpha, \beta, \gamma]\rho^{-1} = [\gamma, \alpha, \beta] \in T$. Moreover, $\tau[\alpha, \beta, \gamma]\tau = [\beta, \alpha, \gamma] \in T$.

COROLLARY. *If a quasigroup Q_1 is carried into a quasigroup Q_2 by a parastrophism π, then every quasigroup isotopic to Q_1 is carried by π into an isotope of Q_2.*

PROOF. Let $Q_1\kappa$ be the isotope of Q_1. We have $Q_2 = Q_1\pi$, hence $Q_1\kappa\pi = Q_2\pi^{-1}\kappa\pi = Q_2\lambda$, with $\lambda = \pi^{-1}\kappa\pi \in T$, by Theorem 1.

The Corollary provides a theoretical basis for the practical rules by which Latin squares of a given order were classified and tabulated by Fisher and Yates [4] and which were followed in the later publica-

tions in this field. The procedure started with the determination of isotopy classes. Then for every isotopy class of Latin squares, i.e., Cayley tables of quasigroups, the ≤ 6 parastrophic isotopy classes were determined. For instance, in the case of quasigroups of order 6, there are 22 distinct isotopy classes, which appear in 5 parastrophic triples, and 7 self-parastrophic isotopy classes. Parastrophic sextuples of isotopy classes were found for quasigroups of order 7.

2. If the quasigroup is a loop, i.e., has a two-sided unit element, a related concept has been introduced. In a loop $(Q, \cdot)$ let the permutations $L(x)$ and $R(y)$ be defined by $xR(y) = yL(x) = xy$, and xJ and xJ^{-1} by $x \cdot xJ = xJ^{-1} \cdot x = 1$, the unit element, for all x and y in Q. Now define $x(\sigma)y = yR^{-1}(xJ)$. With the operation (σ), Q is again a loop, (Q, σ). It is shown in [2] and can be easily ascertained directly that $x(\sigma^{-1})y = xL^{-1}(yJ^{-1})$ and that (Q, σ^{-1}) is a loop. These loops have been studied for loop-theoretic reasons [1], as well as for considerations in geometry [2]. The mapping that takes $(Q, \cdot)$ into (Q, σ) will be called σ. The mapping τ, as defined above, preserves the unit element, hence maps loops on loops. Every product of a finite number of σ, σ^{-1} and τ is called an *isostrophism*. The isostrophisms form a group, R, which by [1] is the infinite dihedral group or one of its homomorphs.

An isotopism mapping a loop onto another loop is called a *loop-isotopism*. The product of a finite number of loop-isotopisms and isostrophisms will be called a *motion*. The use of quasigroups, and in particular of Theorem 1, provides us with a simple proof of the following theorem, proved in [2] in a purely loop-theoretic way.

THEOREM 2.1. *If a loop-isotopism is transformed by a motion, the result is a loop-isotopism.*

PROOF. If κ is a loop-isotopism, we have to prove that $\sigma^{-1}\kappa\sigma$, $\sigma\kappa\sigma^{-1}$ and $\tau\kappa\tau$ are loop-isotopisms. Theorem 1 yields at once $\tau\kappa\tau \in T$. Since both τ and κ preserve the loop property, $\tau\kappa\tau$ is a loop-isotopism. Now, $x(\sigma)y = yR^{-1}(xJ)$ can be represented as a product of two mappings, namely, the parastrophism $\rho: (x, y, z) \to (y, z, x)$, post-multiplied by the isotopism $\iota = [J^{-1}, I, I]$. Indeed, the first step yields $w(\rho)(vw) = v$, and with $vw = y$, $w(\rho)y = yR^{-1}(w)$. Then $(wJ^{-1})(\rho\iota)y = yR^{-1}(w)$, or, with $wJ^{-1} = x$, $x(\rho\iota)y = yR^{-1}(xJ) = x(\sigma)y$. Thus, if $\kappa = [\alpha, \beta, \gamma]$,

$$\sigma^{-1}\kappa\sigma = (\rho[J^{-1}, I, I])^{-1}[\alpha, \beta, \gamma]\rho[J^{-1}, I, I]$$
$$= [J^{-1}, I, I]^{-1}\rho^{-1}[\alpha, \beta, \gamma]\rho[J^{-1}, I, I]$$
$$= [J, I, I][\beta, \gamma, \alpha][J^{-1}, I, I] = [J\beta J^{-1}, \gamma, \alpha] \in T.$$

This is a loop-isotopism because κ is a loop-isotopism and because σ and σ^{-1} carry loops into loops. Similarly,

$$\sigma\kappa\sigma^{-1} = \rho[J^{-1}, I, I][\alpha, \beta, \gamma][J, I, I]\rho^{-1} = \rho[J^{-1}\alpha J, \beta, \gamma]\rho^{-1}$$
$$= [\gamma, J^{-1}\alpha J, \beta],$$

a loop-isotopism. This completes the proof.

The following theorem clarifies the relation between isostrophy and parastrophy.

THEOREM 2.2. *Two isotopy classes $\mathfrak{C}_1$ and $\mathfrak{C}_2$ of quasigroups are parastrophic if and only if there exist two loops $Q_1 \in \mathfrak{C}_1$ and $Q_2 \in \mathfrak{C}_2$ which are isostrophic.*

PROOF. Every isostrophism can be written in the form $\sigma^n\tau^e$, n an integer, $e = 0$ or 1 [1]. Let $Q_1 \in \mathfrak{C}_1$ and $Q_2 \in \mathfrak{C}_2$ be loops, and $Q_2 = Q_1\sigma^n\tau^e = Q_1(\rho[J^{-1}, I, I])^n\tau^e$. Theorem 1 implies that for every isotopism κ, $\rho\kappa = \lambda\rho$, λ also an isotopism. Thus the ρ's can be shifted successively to the right to yield $Q_2 = Q_1\mu\rho^n\tau^e$, μ an isotopism. Hence Q_2 is parastrophic to $Q_1\mu$, that is, $\mathfrak{C}_2$ parastrophic to $\mathfrak{C}_1$. Conversely, let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two parastrophic isotopy classes of quasigroups. Each isotopy class is well known [3] to contain at least one loop. If $\mathfrak{C}_2 = \mathfrak{C}_1\rho$, and $Q_1 \in \mathfrak{C}_1$ and $Q_3 \in \mathfrak{C}_2$ are loops, then there is an isotopism κ such that $Q_3\kappa = Q_1\rho = Q_1\sigma[J, I, I]$, $Q_1\sigma = Q_3\kappa[J^{-1}, I, I]$, that is, the loop $Q_2 = Q_3\kappa[J^{-1}, I, I] \in \mathfrak{C}_2$ is isostrophic to $Q_1 \in \mathfrak{C}_1$. If $\mathfrak{C}_2 = \mathfrak{C}_1\rho^2$, then $\mathfrak{C}_2\rho = \mathfrak{C}_1$, and the argument can be repeated. If $\mathfrak{C}_2 = \mathfrak{C}_1\rho\tau$, we have $Q_3\kappa = Q_1\sigma[J, I, I]\tau = Q_1\sigma\tau[I, J, I]$ and $Q_1\sigma\tau = Q_3\kappa[I, J^{-1}, I] = Q_2$, a loop $\in \mathfrak{C}_2$. If $\mathfrak{C}_2 = \mathfrak{C}_1\tau\sigma$, then $Q_3\kappa = Q_1\tau\sigma[J, I, I]$, and $Q_1\tau\sigma = Q_3\kappa[J^{-1}, I, I] = Q_2$, a loop $\in \mathfrak{C}_2$. The cases $\mathfrak{C}_1\tau = \mathfrak{C}_2$ and $\mathfrak{C}_1 = \mathfrak{C}_2$ are trivial.

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RUTGERS, THE STATE UNIVERSITY