ON REGULAR LOCAL RINGS[1]

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This paper generalizes slightly a result of Kunz [1] and Nakai [2]. If $R > S$ are commutative rings with identity we introduce a module $D^*(R/S)$ defined as the quotient of the module $D(R/S)$ of S differentials of R by the submodule consisting of elements which are mapped to zero by every homomorphism of $D(R/S)$ having values in a finitely generated R module. The characteristic exponent of a field is defined to be 1 if the field is of characteristic zero and to be p if the characteristic of the field is p. The result is then: If R is a local ring containing a field k of characteristic exponent p such that $D^*(R/k^p)$ is finitely generated, then the following conditions are equivalent: (i) R is a regular local ring. (ii) $D^*(R/k^p)$ is free and if x is an element of the completion of R such that $x^p = 0$ then $x = 0$. (iii) $D^*(R/k^p)$ is free and if x is an element of the form ring of R such that $x^p = 0$ then $x = 0$. We remark that in characteristic zero regularity (under the finiteness condition) is equivalent to the freedom of $D^*(R/k)$ and in any case if the local ring is of the form A_q where A is a finitely generated integral domain and q is a prime the second part of (ii) is automatically satisfied. (See Zariski and Samuel [4, p. 314].)

LEMMA 1. *If $R > S$ are commutative rings with identity then there is one and only one module $D^*(R/S)$ (to within R-isomorphism) satisfying the conditions*: (i) *There is an S-derivation d^* from R to $D^*(R/S)$ such that the image of d^* generates $D^*(R/S)$.* (ii) *If h is an S derivation from R to a finitely generated R module M then there is an R homomorphism $D^*(h)$ from $D^*(R/S)$ to M such that $D^*(h)d^* = h$.* (iii) *If f is an element of $D^*(R/S)$ then $h(f) = 0$ for every homomorphism h of D^* to a finitely generated R module if and only if $f = 0$.*

PROOF. If $F(R/S)$ denotes the collection of elements of $D(R/S)$ annihilated by all R homomorphisms to finitely generated R modules, let q denote the quotient map from $D(R/S)$ to $D(R/S)/F(R/S) = D^*(R/S)$ and set $d^* = qd$ where d is the derivation from R to $D(R/S)$. If h is an S derivation from R to a module N denote by $D(h)$ the homomorphism from $D(R/S)$ satisfying $D(h)d = h$ and suppose M is a second module with properties (i)–(iii) where $d\#$ denotes the derivation from R to M and $D\#(h)$ denotes the homomorphism assigned to a derivation from R to N. If b maps M to a finitely

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generated module then one checks easily that $D(b\ d\#)=bD(d\#)$ and hence if f is an element of $F(R/S)$ we have $D(d\#)(f)=0$. Denote by $D^*(d\#)$ the homomorphism from $D^*(R/S)$ to M satisfying the equation $D^*(d\#)q(f)=D(d\#)(f)$ for f in R. If x is an element of $D(R/S)$ with $D^*(d\#)q(x)=0$ then for a homomorphism g from $D(R/S)$ to a finitely generated module P it follows that $D\#(g\ d)\ D(d\#)=g$, thus $g(x)=0$ and $D^*(d\#)$ is an isomorphism.

LEMMA 2. *If R is a local ring containing a field K such that $D^*(R/K)$ is finitely generated, or if $M=D(R/K)/\bigcap m^nD(R/K)$ is finitely generated then $M=D^*(R/K)$ (m the maximal ideal of R).*

PROOF. First $D^*(R/K)$ is clearly hausdorff. Denote by h the quotient map from R to $R/m=L$ and note that $D^*(L/K)=D(L/K)$, using Lemma 1. Let A be the submodule of $R/m\otimes D^*(R/K)$ generated by the elements of the form $1\otimes d^*x$ for x in m and set $D\#(L/K)=[R/m\otimes D^*(R/K)]/A$. Define a derivation $d\#$ from R/m to $D\#(L/K)$ by $d\#(x)=\mathrm{Cl}(1\otimes d^*x)$ where $\mathrm{Cl}(y)$ denotes the coset determined by the element y. If f is a derivation (over K) from R/m to M, a finitely generated L module, then fh is a derivation of R to M and the map $1\otimes D^*(fh)$ induces a homomorphism $D\#(f)$ from $D\#(L/K)$ to M such that $D\#(f)d\#(x)=f(x)$. Since $D\#(L/K)$ is finitely generated it satisfies (i)–(iii) of Lemma 1 and thus $D\#(L/K)=D^*(L/K)=D(L/K)$. Denote by $R(n)$ the ring R/m^n and by $m(n)$ the maximal ideal of $R(n)$ and note that we have the exact sequence $m(n)/m(n)^2\rightarrow R(n)/m(n)\otimes D(R(n)/K)\rightarrow D(L/K)\rightarrow 0$ (Nakai [2, Proposition 9]). It follows easily that $D(R(n)/K)$ is finitely generated. Now consider an element z of $F(R/K)$ and note that if $h(n)$ denotes the quotient map from R to $R(n)$ and if $d(n)$ is the K derivation from $R(n)$ to $D(R(n)/K)$ then $D(d(n)h(n))$ is a homomorphism from $D(R/K)$ to $D(R(n)/K)$ with kernel contained in $m^{n-1}D(R/K)$ (Nakai [2, Proposition 9]) and thus z is an element of $\bigcap m^nD(R/K)$. If M is finitely generated then one checks easily that it satisfies conditions (i)–(iii) of Lemma 1.

If R is a local ring with maximal ideal m and M is a hausdorff R module in the m-adic topology denote by $\mathrm{Co}(M)$ the completion of M.

LEMMA 3. *If R is a local ring containing a field K with $D^*(R/K)$ finitely generated then the completion of $D^*(R/K)$ is isomorphic to $D^*(\mathrm{Co}(R)/K)$.*

PROOF. It will suffice to show that there is an isomorphism from the module $D^*(R/K)/m^nD^*(R/K)$ to $D^*(\mathrm{Co}(R)/K)/m^nD^*(\mathrm{Co}(R)/K)$ which commutes with the quotient maps. Denote by $D(n)^*$ the module $D^*(R/K)/m^nD^*(R/K)$ and by $C(n)^*$ the module $D^*(\mathrm{Co}(R)/K)/m^nD^*(\mathrm{Co}(R)/K)$. Let $p(n+1/n)$ and $q(n+1/n)$ represent the maps from $D(n+1)^*$ to $D(n)^*$ and from $C(n+1)^*$ to $C(n)^*$ respectively and denote by $p(n)$ and $q(n)$ the quotient maps from $D^*(R/K)$ to $D(n)^*$ and from $D^*(\mathrm{Co}(R)/K)$ to $C(n)^*$. If c^* denotes the derivation from $\mathrm{Co}(R)$ to $D^*(\mathrm{Co}(R)/K)$ then the derivation $d(n)\# = q(n)c^*$ gives rise to a homomorphism $D(d(n)\#)$ from $D(R/K)$ to $C(n)^*$ such that $D(d(n)\#)d = d(n)\#$. If f is an element of $m^nD(R/K)$ then $D(d(n)\#)(f) = 0$ and thus there is a homomorphism $J(n)$ from $D(n)^*$ to $C(n)^*$ satisfying the equation $(J(n)p(n))(r(g)) = D(d(n)\#)(g)$ for g in $D(R/K)$ and r the quotient map from $D(R/K)$ to $D^*(R/K)$. Suppose h is in $\mathrm{Co}(R)$, write h in the form $h = h' + h''$ with h' in R and h'' in $m^{n+1}\mathrm{Co}(R)$ and set $e(n)[h] = q(n)d^*(h')$. The map $e(n)$ defines a K derivation from $\mathrm{Co}(R)$ to $D(n)^*$ where we consider $D(n)^*$ as a $\mathrm{Co}(R)$ module by setting $h \cdot u = h'$ for h' as above. Denote by $C(v)$ the homomorphism from $D(\mathrm{Co}(R)/K)$ to P determined if v is a K derivation from $\mathrm{Co}(R)$ to P. If u is an element of $m^nD(\mathrm{Co}(R)/K)$ then $C(e(n))(u)$ is an element of $m^nD(R/K)$ whence there is a homomorphism $H(n)$ from $C(n)^*$ to $D(n)^*$ with $H(n)(q(n)s(f)) = C(e(n))(f)$ for s the quotient map from $D(\mathrm{Co}(R)/K)$ to $D^*(\mathrm{Co}(R)/K)$. For x in R we have $(H(n)\ J(n))(p(n)d^*x) = p(n)d^*x$ and hence $J(n)$ is a monomorphism. If x is in $\mathrm{Co}(R)$ then writing $x = x' + x''$ with x'' in $m^{n+1}\mathrm{Co}(R)$ and x' in R we have that $c^*(x) = c^*(x')$ modulo $m^nD^*(\mathrm{Co}(R)/K)$ from which it follows that $J(n)$ is onto. To complete the assertion we need only show that $J(n)\ p(n+1/n) = q(n+1/n)J(n+1)$ and it suffices to show this for the elements of the form $p(n+1)rdx$ which one checks easily.

As a consequence we have that if $R = K[[X_1, \cdots, X_n]]$ with $[K; K^p] < \infty$ then $D^*(R/K^p)$ is free on the basis d^*X_i and d^*Y_j where Y_j is a p basis of K over K^p. This follows by completing $K[X_1, \cdots, X_n]_X$ where X is the ideal generated by the X_i. Also note that if R is any local ring containing a field K such that $D^*(R/K)$ is finitely generated and if M is a hausdorff R module then any homomorphism from $D(R/K)$ to M annihilates $F(R/K)$.

PROPOSITION 1. *Let R be a local ring containing a field K such that $D^*(R/K)$ is finitely generated. If $R' > R$ with R' regularly quasi-finite over R then $D^*(R'/K)$ is finitely generated.*

(For definitions see [3]).

PROOF. Suppose $R' = R[x_1, \cdots, x_n]$ and assume $R' = R''_{m''}$ where m'' is a maximal ideal of R''. Denote by N'' the image of R'' under the map $d \cdot k$ where k is the inclusion of R'' into R' and let g denote the induced map from $D(R''/K)$ to $D^*(R'/K)$. The image of g is spanned by the restriction of g to the set $(d\cdot k)(R)$ and by the $g\,dx_i$. We note first that $D^*(R'/K)$ is a hausdorff R' module and thus is a hausdorff R module. The map $g(d\cdot k)$ restricted to R is thus a K derivation of R to a hausdorff R module and hence the image of g is generated by a homomorphic image of $D^*(R/K)$ and by the $g\,dx_i$ whence N'' is finitely generated. Now suppose f is in R'. There is an element n of R'', n not in m'', such that nf lies in R'', thus $d^*(nf) = d^*(n)\cdot f + nd^*(f)$, hence $d^*(f) = (1/n)\cdot h$ where h is in the image of g so $D^*(R'/K)$ is finitely generated.

Lemma 4. *Suppose f is an epimorphism of the local ring R to the local ring R' such that R contains a field K with $D^*(R/K)$ and $D^*(R'/K)$ finitely generated. If $A =$ kernel (f) then we have the exact sequence* $(R/A)\otimes A \to R/A \otimes D^*(R/K) \to D^*(R'/K) \to 0$.

Proof. Denote by B the submodule of $R/A \otimes D^*(R/K)$ generated by the elements of the form $1\otimes d^*a$ where a is in A and set $M = [R/A \otimes D^*(R/K)]/B$. If h is the quotient map of $R/A \otimes D^*(R/K)$ onto M we set $g(x) = h(1\otimes d^*x')$ where $f(x') = x$ and note that this defines a map of R/A into M which is independent of the representation x' and is a K derivation of R/A. The induced homomorphism $H^* = D^*(g)$ from $D^*(R'/K)$ to M is such that $H^*(d^*x) = g(x)$ for x in R/A. On the other hand the module $D^*(R'/K)$ is finitely generated as an R module and the map d'^*f from R to $D^*(R'/K)$ is a K derivation of R, thus $D^*(d'^*f)$ maps $D^*(R/K)$ to $D^*(R'/K)$ such that $D^*(d'^*f)(d^*x) = d'^*f(x)$ for x in R. It follows that $1\otimes D^*$ carries M into $D^*(R'/K)$ by $(1\otimes D^*)(x\otimes y) = xD^*(y)$ with $(1\otimes D^*)(dz) = 0$ for z in A. There is thus a map E^* from M to $D^*(R'/K)$ and one need only check that H^*E^* and E^*H^* are the identity.

Lemma 5. *Let $R = K[[X_1, \cdots, X_n]]$ with $[K; K^p] < \infty$ where p is the characteristic exponent of K, suppose A is an ideal of R and assume that $D(A) < A$ for every K^p derivation of R into R. If $A \neq 0$ then* (i) *there is an element x of R such that x is not in A but x^p is in A, or $A = R$ and* (ii) *there is an element x of the form ring of R/A with $x \neq 0$ and $x^p = 0$ or $A = R$.*

Proof. Choose a p-basis for K over K^p say $y_1, \cdots, y_r$. If Q is a power series in R we define the total degree of Q to be the pair (u, v) where u is the subdegree of Q and v is the degree of the leading form of Q considered as a polynomial in the y_j. Order the total degrees lexicographically and choose an element P of A of least total degree (a, b) and assume $b \neq 0$. Since b is nonzero the partial derivative of P

with respect to one of the y_j occurring in the leading form $L(P)$ of P lies in A and reduces the total degree, thus no y_j may occur in $L(P)$ and $L(P)$ is in $K^p[X_1, \cdots, X_n]$. Now consider any one of the indeterminates X_i and note that the subdegree of P will be reduced by differentiating with respect to X_i unless the exponent to which X_i occurs in a given monomial of $L(P)$ is of the form sp. We therefore have assertion (ii). If Q is in R we may write it in the form $Q=\sum Q_a$ where a ranges over the collection of all the subsets of $T=\langle 1, \cdots, n\rangle$ (including the empty set) and Q_a is the sum of those monomials M of Q such that X_i appears in M with exponent of the form sp for those and only those i in a. We now denote by B_j the operation $X_j\partial/\partial X_j$ and note that B_j maps A into itself and that B_j is zero on the monomials of A in which X_j occurs with an exponent which is a multiple of p. The application of B_j $(p-1)$ times is the identity on any monomial which does not have X_j occurring with exponent a multiple of p. If Q is in A then applying B_n $(p-1)$ times and subtracting the result from Q we have that $\sum Q_a$ is in A where the sum runs over those subsets of T which contain n and by induction we have that Q_T is in A. Using (i) we have that there are elements of A such that $Q_T\neq 0$. Let W denote the collection of elements of A of least subdegree which lie in $K[[X_1^p, \cdots, X_n^p]]$ and let t be the least degree of the elements of W considered as polynomials in the y_j with coefficients in $K^p[[X_1^p, \cdots, X_n^p]]$. To prove assertion (i) it suffices to show that $t=0$. Note, however, that the set W remains fixed under the partial derivatives with respect to the y_j from which the assertion is immediate.

THEOREM. *If R is a local ring containing a field k of characteristic exponent p such that $D^*(R/k^p)$ is finitely generated then the following conditions are equivalent*: (i) *R is regular*, (ii) *$D^*(R/k^p)$ is free and if x is an element of* Co(R) *such that $x^p=0$ then $x=0$*, (iii) *$D^*(R/k^p)$ is free and if x is an element of the form ring of R such that $x^p=0$ then $x=0$.*

PROOF. We first remark that if $k<K$ then in characteristic p (nonzero) we have that $D^*(R/k^p)=D^*(R/K^p)$ and in any case by Lemma 3 we may suppose that the ring R is complete. We may therefore assume in nonzero characteristic that $k=K$ is a field of coefficients of R. Consider the map $1\otimes d\#$ carrying the module m/m^2 into $R/m\otimes D^*(R/K)$. To prove $1\otimes d\#$ is an injection it suffices to prove the assertion for $R/m^2=R^*=K+m^*$ where m^* is the maximal ideal of R^*. The projection g of R^* onto m^* is a K derivation to a finitely generated R^* module thus $D^*(g)$ maps $D^*(R/K)$ to m^* such that $D^*(g)d\#m=g(m)=m$ for m in m^*, thus $1\otimes d\#$ is an injection. The map

$1\otimes d^*$ from m/m^2 to $R/m\otimes D^*(R/K^p)$ is such that if x is in m and $1\otimes d^*(x)=0$ then $1\otimes D^*(d\#)(1\otimes d^*)(x)=0$ which implies x is in m^2 by the above. Thus we have an exact sequence $0\rightarrow m/m^2 \rightarrow R/m\otimes D^*(R/K^p)\rightarrow D^*(K/K^p)\rightarrow 0$ in nonzero characteristic. Similarly in zero characteristic we may replace K^p by k in the above sequence. In nonzero characteristic we have that $[K:K^p]<\infty$ and a basis for $D(K/K^p)$ is given by the dY_j where the Y_j are a p-basis for K over K^p, and thus if m_i, $1\leqq i\leqq n$ is a minimal system of generators for the maximal ideal of R the elements d^*m_j and d^*Y_i are a basis for the module $D^*(R/K^p)$. In characteristic zero the d^*m_j are a subset of a basis for $D^*(R/k)$. Let f be a map from $K[[X_1, \cdots, X_n]]$ onto R carrying K onto K and X_i onto m_i where the X_i are indeterminates. Set $N=\text{kernel } (f)$ and assume x is in N. We have the equation $0=d^*(f(x))=\sum f(\partial x/\partial X_i)d^*m_i+\sum f(\partial x/\partial Y_j)d^*Y_j$ in characteristic $p\neq 0$ and since $D^*(R/K^p)$ is free all the partials of x must be in N which implies $N=0$ under the assumptions (ii) and (iii) by Lemma 5. In the case of characteristic zero we have that the partials with respect to the X_i all are in N since the d^*m_i can be extended to a basis and we may again apply Lemma 5.

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