THE RADIUS OF UNIVALENCE OF CERTAIN
ANALYTIC FUNCTIONS

THOMAS H. MACGREGOR

1. Introduction. Suppose that \( f(z) = z + a_2 z^2 + \cdots \) is analytic for \( |z| < 1 \). If \( \text{Re}\{f(z)/z\} > 0 \) for \( |z| < 1 \) then \( f(z) \) is univalent in \( |z| < \sqrt{2} - 1 \) [5, Theorem 3; 7]. The function \( f(z) = (z + z^2)/(1 - z) \) satisfies the hypotheses but is univalent in no circle \( |z| < r \) for \( r > \sqrt{2} - 1 \) since its derivative vanishes at \( z = \sqrt{2} - 1 \).

In this paper we generalize the above theorem for functions whose power series begins \( f(z) = z + a_n z^{n+1} + \cdots \). The estimate used to obtain this result is further used to find the radius of convexity for functions \( f(z) = z + a_{n+1} z^{n+1} + \cdots \) which are analytic and satisfy \( \text{Re} f'(z) > 0 \) for \( |z| < 1 \). For \( n = 1 \) this theorem is not new [5, Theorem 2; 10, p. 284]. The condition \( \text{Re} f'(z) > 0 \) is known to be sufficient for the univalency of \( f(z) \) in \( |z| < 1 \) [1, p. 18].

We consider the problem of finding the radius of univalence for functions \( f(z) = z + a_2 z^2 + \cdots \) which are analytic and satisfy \( \text{Re}\{f(z)/g(z)\} > 0 \) for \( |z| < 1 \), where \( g(z) = z + b_2 z^2 + \cdots \) is analytic and univalent for \( |z| < 1 \). In the case that \( g(z) \) is either starlike or convex this problem is solved. We take particular advantage of the condition \( \text{Re}\{z f'(z)/f(z)\} > 0 \) for \( |z| < r \), which is necessary and sufficient for \( f(z) \) to be univalent and starlike in \( |z| < r \) [8, p. 105, problem 109]. For arbitrary univalent functions \( g(z) \) we only obtain an estimate for the radius of univalence for \( f(z) \).

2. Lemma 1. Suppose that \( h(z) = 1 + c_n z^n + \cdots \) is analytic and satisfies \( \text{Re} h(z) > 0 \) for \( |z| < 1 \). Then

\[
\frac{|h'(z)|}{h(z)} \leq \frac{2n}{1 - |z|^{2n}}.
\]

Proof. Let \( k(z) = (1 - h(z))/(1 + h(z)) = d_n z^n + \cdots \). Then \( k(z) \) is analytic for \( |z| < 1 \) and \( |k(z)| < 1 \). Thus, \( k(z) = z^n \phi(z) \) where \( \phi(z) \) is analytic for \( |z| < 1 \) and \( |\phi(z)| \leq 1 \). For such functions we have

\[
|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}
\]

[2, p. 18].

Expressing \( h(z) \) and \( h'(z) \) in terms of \( \phi(z) \) gives

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Using (1) we obtain
\[
\left| \frac{h'(z)}{h(z)} \right| \leq \frac{2 |z|^{n-1} \left| (1 - |\phi(z)|^2) + n(1 - |z|^2) |\phi(z)| \right|}{1 - |z|^{2n} |\phi(z)|^2}.
\]

To prove the lemma it is sufficient to show that for \(|z| = r, 0 < r < 1\),
\[
\frac{r(1 - |\phi(z)|^2) + n(1 - r^2) |\phi(z)|}{1 - r^{2n} |\phi(z)|^2} \leq \frac{n(1 - r^2)}{1 - r^{2n}}.
\]

Letting \(x = |\phi(z)|\) this is equivalent to \((1 - x) F_n(x) \geq 0\) for \(0 \leq x \leq 1\), where
\[
F_n(x) = a - bx, \quad a = n(1 - r^2) - r(1 - r^{2n}) > (1 - r^2)(n - nr) > 0,
\]
\[
b = r(1 - r^{2n}) - nr^{2n}(1 - r^2)
\]
\[
= r(1 - r^2)(1 + r^2 + r^4 + \cdots + r^{2n-2} - nr^{2n-1})
\]
\[
= r(1 - r^2) \{|1 - r^{2n-1}| + (r^2 - r^{2n-1}) + \cdots + (r^{2n-2} - r^{2n-1})\} > 0.
\]

Since \(F_n(x) \geq F_n(1)\) we can prove \(F_n(x) \geq 0\) by showing that \(F_{n+1}(1) \geq F_n(1)\) and \(F_1(1) \geq 0\).
\[
F_{n+1}(1) - F_n(1)
\]
\[
= (1 - r^2)(1 - r^{2n+1} - r^{2n+2} - nr^{2n+1} + r^{2n+2})
\]
\[
= (1 - r^2)(1 - r)\{|1 + r + r^2 + \cdots + r^{2n} - r^{2n+1} - nr^{2n} + nr^{2n+1}\}
\]
\[
> 0.
\]

This inequality follows since the negative terms in the brackets can be expressed as \(2n + 1\) terms each of which is numerically less than a corresponding positive term.

Finally, \(F_1(1) = (1+r)(1-r)^2 > 0\).

One can show that the equality holds in the lemma only for the functions \(h(z) = (1 - ez^n)/(1 + ez^n)\) where \(|e| = 1\) and for appropriate values of \(z\).

**Theorem 1.** Suppose that \(f(z) = z + a_{n+1}z^{n+1} + \cdots\) is analytic and

\[1\] I would like to thank the referee of this paper for simplifying my argument for the remaining part of the proof.
satisfies $\text{Re}\{f(z)/z\} > 0$ for $|z| < 1$. Then $f(z)$ is univalent and starlike in $|z| < ((n^2 + 1)^{1/2} - \pi)^{1/n}$.

**Proof.** Since $\text{Re}\{f(z)/z\} > 0$ we can infer that $f(z)$ cannot vanish in $|z| < 1$ except for a simple zero at $z = 0$. Let

$$h(z) = \frac{f(z)}{z} = 1 + a_{n+1}z^{n+1} + \cdots, \quad \text{Re} h(z) > 0 \quad \text{for } |z| < 1.$$

From Lemma 1 we have

$$\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2n|z|^n}{1 - |z|^{2n}}.$$

Also

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zh'(z)}{h(z)}.$$

Therefore, $f(z)$ will be univalent and starlike if $|zh'(z)/h(z)| < 1$. From the above estimate this is satisfied if $(2n|z|^n)/(1 - |z|^{2n}) < 1$, i.e., for $|z| < ((n^2 + 1)^{1/2} - \pi)^{1/n}$.

The function $f(z) = (z + zn+1)/(1 - zn) = z + \frac{2zn+1}{n+1} + \cdots$ satisfies

$$\text{Re}\left\{\frac{f(z)}{z}\right\} > 0 \quad \text{for } |z| < 1$$

but is not univalent in $|z| < r$ for $r > r_n = ((n^2 + 1)^{1/2} - \pi)^{1/n}$ since $f'(r\cos(n\pi/n)) = 0$.

**Theorem 2.** Suppose that $f(z) = z + a_{n+1}z^{n+1} + \cdots$ is analytic and satisfies $\text{Re} f'(z) > 0$ for $|z| < 1$. Then $f(z)$ is convex in $|z| < ((n^2 + 1)^{1/2} - n)^{1/n}$.

**Proof.** We can apply Lemma 1 to $f'(z) = 1 + (n+1)a_{n+1}z^n + \cdots$ since $\text{Re} f'(z) > 0$. This gives

$$|f''(z)/f'(z)| \leq 2n|z|^{n-1}/(1 - |z|^{2n}).$$

The condition $\text{Re}\{(zf''(z)/f'(z)) + 1\} > 0$ for $|z| < r$ is necessary and sufficient for $f(z)$ to map $|z| < r$ onto a convex domain (Problem 108, p. 105). This condition is satisfied if $|zf''(z)/f'(z)| < 1$. From the above estimate we can deduce that $f(z)$ is convex if $(2n|z|^n)/(1 - |z|^{2n}) < 1$. This inequality is equivalent to $|z| < ((n^2 + 1)^{1/2} - n)^{1/n}$.

The function

$$f(z) = \int_0^z \frac{1 + \sigma^n}{1 - \sigma^n} d\sigma = z + \frac{2}{n+1} z^{n+1} + \cdots$$

is an extremal function for Theorem 2.
3. **Theorem 3.** Suppose that \( f(z) = z + a_2z^2 + \cdots \) and \( g(z) = z + b_2z^2 + \cdots \) are analytic for \( |z| < 1 \) and \( g(z) \) is univalent and starlike for \( |z| < 1 \). If \( \text{Re} \{ f(z)/g(z) \} > 0 \) for \( |z| < 1 \) then \( f(z) \) is univalent and starlike in \( |z| < 2 - \sqrt{3} \).

**Proof.** The hypotheses imply that \( f(z) \) and \( g(z) \) do not vanish in \( |z| < 1 \) except for the simple zero at \( z = 0 \). Let

\[
\frac{h(z)}{g(z)} = 1 + c_1z + \cdots, \quad \text{Re} \ h(z) > 0 \quad \text{for} \quad |z| < 1.
\]

Applying Lemma 1 to \( h(z) \) for \( n = 1 \) gives \( |zh'(z)/h(z)| \leq 2|z|/(1 - |z|^2) \). Since \( g(z) \) is starlike \( \text{Re} \{ zg'(z)/g(z) \} > 0 \) for \( |z| < 1 \). Thus \( \text{Re} \{ zg'(z)/g(z) \} \geq (1 - |z|)/(1 + |z|) \) [8, problem 287, p. 140].

\[
\frac{zf''(z)}{f(z)} = \frac{sg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}
\]

\[
\text{Re} \left\{ \frac{zf''(z)}{f(z)} \right\} \geq \text{Re} \left\{ \frac{sg'(z)}{g(z)} \right\} - \left| \frac{zh'(z)}{h(z)} \right| \geq \frac{1 - |z|}{1 + |z|} - \frac{2|z|}{1 - |z|^2} = \frac{1 - 4|z| + |z|^2}{1 - |z|^2}.
\]

Thus, \( \text{Re} \{ zf''(z)/f(z) \} > 0 \) if \( 1 - 4|z| + |z|^2 > 0 \). The last inequality is satisfied for \( |z| < 2 - \sqrt{3} \). Therefore \( f(z) \) is univalent in \( |z| < 2 - \sqrt{3} \) and maps that circle onto a starlike domain.

The function \( f(z) = (z + z^2)/(1 - z)^2 \) satisfies the hypotheses of Theorem 3 where \( g(z) = z/(1 - z)^2 \) and \( h(z) = (1 + z)/(1 - z) \). The derivative of this function vanishes at \( z = \sqrt{3} - 2 \). Thus, it is univalent in no circle \( |z| < r \) with \( r > 2 - \sqrt{3} \).

For a part of the next theorem we need a sharpening of Lemma 1 for \( n = 1 \). This result is known but we give a short proof of it here.

**Lemma 2.** Suppose that \( h(z) = 1 + c_1z + \cdots \) is analytic and satisfies \( \text{Re} \ h(z) > 0 \) for \( |z| < 1 \). Then \( \text{Re} \ |h'(z)| \leq 2 \text{Re} \ h(z)/(1 - |z|^2) \).

**Proof:** Let \( \phi(z) = (1 - h(z))/(1 + h(z)) \), \( \phi(z) < 1 \) for \( |z| < 1 \). Using

\[
|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}
\]

(1)

\[\text{gives}\]

\[\text{gives}\]
The lemma follows by noting that \( |1 + h(z)|^2 - |1 - h(z)|^2 = 4 \text{ Re } h(z) \).

**Theorem 4.** Suppose that \( f(z) = z + az^2 + \cdots \) and \( g(z) = z + bz^2 + \cdots \) are analytic for \( |z| < 1 \) and \( g(z) \) is univalent and convex for \( |z| < 1 \). If \( \text{ Re } \{ f(z)/g(z) \} > 0 \) for \( |z| < 1 \) then \( \text{ Re } \{ f'(z)/g'(z) \} > 0 \) for \( |z| < \frac{1}{3} \). Also, \( f(z) \) is univalent and starlike for \( |z| < \frac{1}{3} \).

**Proof.** The hypotheses imply that \( f(z), g(z) \) and \( g'(z) \) do not vanish in \( |z| < 1 \) except for the simple zeros of \( f(z) \) and \( g(z) \) at \( z = 0 \). Let \( h(z) = f(z)/g(z) = 1 - f(z)z + \cdots \), \( \text{ Re } h(z) > 0 \) for \( |z| < 1 \).

Applying Lemma 2 to \( h(z) \) gives \( |h'(z)| \leq 2 \text{ Re } h(z)/(1 - |z|^2) \).

Since \( g(z) \) is univalent and convex for \( |z| < 1 \) we have \( \text{ Re } \{ zg'(z)/g(z) \} > \frac{1}{2} \) for \( |z| < 1 \) and consequently \( \text{ Re } \{ zg'(z)/g(z) \} \geq (1 + |z|)^{-1} [6; 9] \).

This implies \( |g(z)/g'(z)| \leq |z| (1 + |z|) \).

Thus, for \( |z| < \frac{1}{3} \) \( \text{ Re } \{ f'(z)/g'(z) \} > 0 \). This shows that \( f(z) \) is univalent and close-to-convex for \( |z| < \frac{1}{3} \) [4].

Let us show that \( f(z) \) maps \( |z| < \frac{1}{3} \) onto a starlike domain.

\[
\text{ Re } \left\{ \frac{f'(z)}{f(z)} \right\} \geq \text{ Re } \left\{ \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} \right\} \\
\geq \text{ Re } \left\{ \frac{zg'(z)}{g(z)} \right\} - \frac{|zh'(z)|}{|h(z)|} \\
\geq \frac{1}{1 + |z|} - \frac{2|z|}{1 - |z|^2} = \frac{1 - 3|z|}{1 - |z|^2}.
\]

Thus, for \( |z| < \frac{1}{3} \) \( \text{ Re } \{ zf'(z)/f(z) \} > 0 \). This shows that \( f(z) \) is univalent and starlike in \( |z| < \frac{1}{3} \).

Theorem 4 gives the radius of univalence for the class of functions considered. In order to show this let \( f(z) = (z + z^2)/(1-z)^2 \), \( g(z) \)
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\[ g(z) = \frac{z}{1-z}. \]
Then, \( g(z) \) is univalent and convex for \( |z| < 1 \). Here, \( h(z) = \frac{1+z}{1-z} \) and therefore \( \Re h(z) > 0 \). This function \( f(z) \) is univalent in no circle \( |z| < r \) with \( r > \frac{1}{2} \) since \( f'(\frac{-1}{2}) = 0 \).

**Theorem 5.** Suppose that \( f(z) = z + a_2z^2 + \cdots \) and \( g(z) = z + b_2z^2 + \cdots \) are analytic for \( |z| < 1 \) and \( g(z) \) is univalent in \( |z| < 1 \). If \( \Re \{ f(z)/g(z) \} > 0 \) for \( |z| < 1 \) then \( f(z) \) is univalent in \( |z| < 1/5 \).

**Proof.** Let \( h(z) = f(z)/g(z) = 1 + cz + \cdots \), \( \Re h(z) > 0 \) for \( |z| < 1 \).

To show that \( f(z) \) is univalent in \( |z| \leq r \) it suffices to show that \( f(z) \) is univalent on \( |z| = r \). Let \( z_1 \neq z_2 \), \( |z_1| = |z_2| = r \). Then, \( f(z_1) = f(z_2) \) can be written

\[
\frac{1}{g(z_1)} \frac{g(z_2) - g(z_1)}{z_2 - z_1} = -\frac{1}{h(z_2)} \frac{h(z_2) - h(z_1)}{z_2 - z_1}.
\]

Thus, if

\[
\left| \frac{g(z_2) - g(z_1)}{g(z_1)(z_2 - z_1)} \right| > \left| \frac{h(z_2) - h(z_1)}{h(z_2)(z_2 - z_1)} \right|
\]

then \( f(z) \) is univalent in \( |z| \leq r \).

Let \( k(z) = (1 - h(z))/(1 + h(z)) \), \( |k(z)| < 1 \) for \( |z| < 1 \) and \( k(0) = 0 \). Therefore \( |k'(z)| \leq 1 \) for \( |z| \leq \sqrt{2} - 1 \) [2, p. 19]. From the representation \( k(z_2) - k(z_1) = \int_{z_1}^{z_2} k'(z)dz \) where the path of integration is the line segment from \( z_1 \) to \( z_2 \), the estimate on \( k'(z) \) gives \( |k(z_2) - k(z_1)/(z_2 - z_1)| \leq 1 \) for \( r \leq \sqrt{2} - 1 \). Expressing \( h(z) \) in terms of \( k(z) \) yields

\[
\left| \frac{h(z_2) - h(z_1)}{h(z_2)(z_2 - z_1)} \right| \leq 2 \left( \frac{1}{1 - |z_1|} \right) \left( \frac{1}{1 - |z_2|} \right) \frac{2}{(1 - r)^2}.
\]

Here we have used Schwarz's lemma \( |k(z)| \leq |z| \).

Since \( g(z) = z + b_2z^2 + \cdots \) is analytic and univalent for \( |z| < 1 \)

\[
\left| \frac{g(z_2) - g(z_1)}{z_2 - z_1} \right| \geq \left| g(z_1)g(z_2) \right| \frac{1 - r^2}{r^2} \geq |z|/(1 + |z|)^2
\]

[3]. Using this estimate and the distortion theorem \( |g(z)| \geq |z|/(1 + |z|)^2 \) we obtain
Therefore, \( f(z) \) is univalent in \( |z| \leq r \) if \( r \leq \sqrt{2} - 1 \) and \( (1-r)/(r(1+r)) > 2/(1-r)^2 \). The last inequality is equivalent to \( 1-5r+r^2-r^3 > 0 \). Since the equation \( 1-5r+r^2-r^3 = 0 \) has one positive root \( r_0 \), where \( 0.20 < r_0 < 0.21 \), we can infer that \( f(z) \) is univalent in \( |z| < r_0 \). In particular, \( f(z) \) is univalent in \( |z| < 1/5 \).

The circle \( |z| < r_0 \) is not the circle of univalence for the functions \( f(z) \) which satisfy the hypotheses of Theorem 5. If it were then we must have \( |g(z)| = |z|/(1+|z|)^2 \) for some \( z \). This estimate holds only for the functions \( g(z) = z/(1+\varepsilon z)^2 \) where \( |\varepsilon| = 1 \). Since these functions are starlike for \( |z| < 1 \) Theorem 3 implies that \( f(z) \) would be univalent \( |z| < 2 - \sqrt{3} \). However, \( 2 - \sqrt{3} = 0.267 \cdots > r_0 \).

References


Lafayette College