THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS. II

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1. Introduction. Throughout this paper suppose that $f(z)=z+a_2z^2+\cdots$ is analytic for $|z|<1$ and $g(z)=z+b_2z^2+\cdots$ is analytic and univalent for $|z|<1$. Recently [3] the author considered the problem: what is the radius of univalence of the family of functions $f(z)$ which satisfy $\operatorname{Re}\{f(z)/g(z)\}>0$ for $|z|<1$? This problem was solved for each of the subfamilies associated with the cases $g(z)\equiv z$, $g(z)$ is starlike for $|z|<1$, and $g(z)$ is convex for $|z|<1$. Only an estimate was obtained for the radius of univalence of the whole family.

In this paper we consider functions $f(z)$ which satisfy $|f(z)/g(z)-1|<1$ for $|z|<1$. The radius of univalence of this family of functions is determined. Also, we find the radius of univalence and starlikeness for the subfamilies associated with each of the cases: $g(z)\equiv z$, $g(z)$ is starlike for $|z|<1$, $g(z)$ is convex for $|z|<1$.

2. Theorem 1. *If $f(z)=z+a_2z^2+\cdots$ is analytic and satisfies $|f(z)/z-1|<1$ for $|z|<1$ then $f(z)$ is univalent and starlike for $|z|<1/2$.*

Proof. The function $F(z)=f(z)/z-1$ is analytic and satisfies $|F(z)|<1$ for $|z|<1$ and $F(0)=0$. Thus $F(z)=z\phi(z)$ where $\phi(z)$ is analytic for $|z|<1$ and $|\phi(z)|\leqq 1$. For such functions we have

$$|\phi'(z)|\leqq\frac{1-|\phi(z)|^2}{1-|z|^2}\tag{1}$$

[1, p. 18]. Expressing $f(z)$ and $f'(z)$ in terms of $\phi(z)$, we obtain

$$\frac{zf'(z)}{f(z)}=1+z\frac{\phi(z)+z\phi'(z)}{1+z\phi(z)}\cdot\tag{2}$$

Using (1) we obtain

$$\left|\frac{\phi(z)+z\phi'(z)}{1+z\phi(z)}\right|\leqq\frac{|\phi(z)|+|z|\dfrac{1-|\phi(z)|^2}{1-|z|^2}}{1-|z|\,|\phi(z)|}$$
$$=\frac{|\phi(z)|+|z|}{1-|z|^2}\leqq\frac{1+|z|}{1-|z|^2}=\frac{1}{1-|z|},$$

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$$\left|\frac{\phi(z)+z\phi'(z)}{1+z\phi(z)}\right| \leqq \frac{1}{1-|z|}. \tag{3}$$

From (2) and (3) we obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqq \frac{|z|}{1-|z|}.$$

If $|z| < 1/2$ then $|zf'(z)/f(z) - 1| < 1$ so that, in particular, $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < 1/2$. This proves the theorem since the condition $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < r$ is necessary and sufficient for $f(z)$ to be univalent for $|z| < r$ and map that circle onto a domain which is starlike with respect to the origin.

The function $f(z) = z + z^2$ satisfies the hypotheses of this theorem but is univalent in no circle $|z| < r$ with $r > 1/2$ since its derivative vanishes at $z = -1/2$.

THEOREM 2. *Suppose that* $f(z) = z + a_2z^2 + \cdots$ *is analytic for* $|z| < 1$ *and* $g(z) = z + b_2z^2 + \cdots$ *is analytic and univalent for* $|z| < 1$. *If* $|f(z)/g(z) - 1| < 1$ *for* $|z| < 1$ *then* $f(z)$ *is univalent for* $|z| < 1/3$.

PROOF. The function $\phi(z) = f(z)/g(z) - 1$ is analytic for $|z| < 1$ and satisfies $|\phi(z)| < 1$, $\phi(0) = 0$. To show that $f(z)$ is univalent in $|z| \leqq r$ it suffices to prove univalence on $|z| = r$. Let $z_1 \neq z_2$, $|z_1| = |z_2| = r$. Then $f(z_1) = f(z_2)$ can be written

$$\frac{g(z_2) - g(z_1)}{g(z_1)(z_2 - z_1)} = -\frac{\phi(z_2) - \phi(z_1)}{(1 + \phi(z_2))(z_2 - z_1)}.$$

Thus, $f(z)$ will be univalent in $|z| \leqq r$ if we prove

$$\left|\frac{g(z_2) - g(z_1)}{g(z_1)(z_2 - z_1)}\right| > \left|\frac{\phi(z_2) - \phi(z_1)}{(1 + \phi(z_2))(z_2 - z_1)}\right|.$$

Since $|\phi(z)| < 1$ for $|z| < 1$ and $\phi(0) = 0$ we have $|\phi'(z)| \leqq 1$ for $|z| \leqq \sqrt{2} - 1$ [1, p. 19]. From the representation $\phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2}\phi'(z)dz$ where the path of integration is the line segment from z_1 to z_2 the estimate for $|\phi'(z)|$ implies $|\phi(z_2) - \phi(z_1)| \leqq |z_2 - z_1|$ for $r \leqq \sqrt{2} - 1$. From Schwarz's lemma we also have $|\phi(z_2)| \leqq |z_2| = r$. Thus,

$$\left|\frac{\phi(z_2) - \phi(z_1)}{(1 + \phi(z_2))(z_2 - z_1)}\right| \leqq \frac{1}{1 - r} \qquad \text{for } r \leqq \sqrt{2} - 1. \tag{4}$$

Since $g(z) = z + b_2z^2 + \cdots$ is analytic and univalent for $|z| < 1$

$$\left|\frac{g(z_2)-g(z_1)}{z_2-z_1}\right| \geqq |g(z_1)g(z_2)| \frac{1-r^2}{r^2}$$

[2]. Using this estimate and the distortion theorem $|g(z)| \geqq |z|/(1+|z|)^2$ we obtain

$$\left|\frac{g(z_2)-g(z_1)}{g(z_1)(z_2-z_1)}\right| \geqq \frac{1-r}{r(1+r)}. \tag{5}$$

From (4) and (5) it follows that $f(z)$ is univalent in $|z| \leqq r$ if $(1-r)/r(1+r) > 1/(1-r)$ and $r \leqq \sqrt{2}-1$. Since $(1-r)/r(1+r) > 1/(1-r)$ is equivalent to $r<1/3$ we have shown that $f(z)$ is univalent in $|z|<1/3$.

The function $f(z)=(z+z^2)/(1-z)^2$ satisfies the hypotheses of this theorem with $g(z)=z/(1-z)^2$. This $f(z)$ is not univalent in $|z|<r$ if $r>1/3$ since its derivative vanishes at $z=-1/3$.

THEOREM 3. *Suppose that $f(z)=z+a_2z^2+\cdots$ is analytic for $|z|<1$ and $g(z)=z+b_2z^2+\cdots$ is analytic, univalent and starlike for $|z|<1$. If $|f(z)/g(z)-1|<1$ for $|z|<1$ then $f(z)$ is univalent and starlike for $|z|<1/3$.*

PROOF. We can write $f(z)/g(z)-1=z\phi(z)$ where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leqq 1$ for $|z|<1$. Expressing $f(z)$ and $f'(z)$ in terms of $g(z)$ and $\phi(z)$ yields

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + z\,\frac{\phi(z)+z\phi'(z)}{1+z\phi(z)}. \tag{6}$$

($zf'(z)/f(z)$ is analytic in $|z|<1$ since the hypotheses imply that $f(z)$ does not vanish in $|z|<1$ except for the simple zero at $z=0$.)

Since $g(z)$ is starlike $\operatorname{Re}\{zg'(z)/g(z)\}>0$ for $|z|<1$ and, therefore,

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqq \frac{1-|z|}{1+|z|}$$

[5, p. 140]. Applying this estimate and also (3) to (6) gives

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1-|z|}{1+|z|} - \frac{|z|}{1-|z|} = \frac{1-3|z|}{1-|z|^2}.$$

Therefore, if $|z|<1/3$ then $\operatorname{Re}\{zf'(z)/f(z)\}>0$. This proves the theorem.

The function $f(z)=(z+z^2)/(1-z)^2$ is an extremal function for this theorem as well as for Theorem 2 since $g(z)=z/(1-z)^2$ maps $|z|<1$ onto a starlike domain.

THEOREM 4. *Suppose that* $f(z)=z+a_2z^2+\cdots$ *is analytic for* $|z|<1$ *and* $g(z)=z+b_2z^2+\cdots$ *is analytic, univalent and convex for* $|z|<1$. *If* $|f(z)/g(z)-1|<1$ *for* $|z|<1$ *then* $f(z)$ *is univalent and starlike for* $|z|<\sqrt{2}-1$.

PROOF. Since $g(z)$ is univalent and convex for $|z|<1$ we have $\operatorname{Re}\{zg'(z)/g(z)\}>1/2$ for $|z|<1$ and, therefore, $\operatorname{Re}\{zg'(z)/g(z)\} \geqq 1/(1+|z|)$ [**4**; **6**]. Proceeding exactly as in the proof of Theorem 3 this improved estimate on $\operatorname{Re}\{zg'(z)/g(z)\}$ gives

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1}{1+|z|}-\frac{|z|}{1-|z|}=\frac{1-2|z|-|z|^2}{1-|z|^2}.$$

If $|z|<\sqrt{2}-1$ then $1-2|z|-|z|^2>0$. Therefore, $f(z)$ is univalent and starlike in $|z|<\sqrt{2}-1$.

The function $f(z)=(z+z^2)/(1-z)$ satisfies the hypotheses of this theorem with $g(z)=z/(1-z)$. Since the derivative of this $f(z)$ vanishes at $z=1-\sqrt{2}$ it is not univalent in $|z|<r$ if $r>\sqrt{2}-1$.

REFERENCES

1. C. Carathéodory, *Theory of functions*, Vol. 2, Chelsea, New York, 1954.
2. G. M. Golusin, *On distortion theorems and coefficients of univalent functions*, Mat. Sb. (N.S.) **19** (**61**) (1946), 183–202. (Russian, English summary)
3. T. MacGregor, *The radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **14** (1963), 514–520.
4. A. Marx, *Untersuchungen über schlichte Abbildungen*, Math. Ann. **107** (1932), 40–67.
5. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. **1**, Springer, Berlin, 1954.
6. E. Strohhäcker, *Beiträge zur Theorie der schlichten Funktionen*, Math. Z. **37** (1933), 356–380.

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