UPPER AND LOWER BOUNDS OF THE NORM OF SOLUTIONS OF DIFFERENTIAL SYSTEMS

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1. Let I denote the half-line $0 \leqq t < \infty$ and R^n the n-dimensional Euclidean space. We consider the differential system

$$x' = f(t, x); \qquad x(t_0) = x_0, \qquad (t_0 \geqq 0) \tag{1}$$

where x and f are n-dimensional vectors and the function $f(t, x)$ is continuous and defined on the product space $I \times R^n$. Let $|x|$ denote any convenient norm of x.

Let the function $V(t, x) \geqq 0$ be continuous and defined on $I \times R^n$. Suppose further that $V(t, x)$ satisfies a Lipschitz condition in x locally for each $t \in I$ and that $V(t, x) \to \infty$ as $|x| \to \infty$. Then we can prove the following results.

THEOREM 1. *Let the function $W(t, r)$ be continuous and defined for $t \in I$, $r \geqq 0$. Suppose that $r(t)$ is the maximal solution of the differential equation*

$$r' = W(t, r); \qquad r(t_0) = r_0, \tag{2}$$

existing for all t to the right of t_0. Assume that

$$V(t + \lambda^{-1}, x + \lambda^{-1} f(t, x)) \leqq V(t, x) + \lambda^{-1} W(t, V(t, x)) + o(\lambda^{-1}), \tag{3}$$

for each $t \in I$, $x \in R^n$ and for all sufficiently large $\lambda > 0$. Then, if $x(t)$ is any solution of (1) *such that $V(t_0, x_0) \leqq r_0$, $x(t)$ can be continued as far as $r(t)$ exists and*

$$V(t, x(t)) \leqq r(t), \qquad (t \geqq t_0). \tag{4}$$

If $V(t, x(t))$ is regarded as a measure of a solution $x(t)$ of (1), the following result gives a better control than (4).

THEOREM 2. *Suppose that the assumptions of Theorem* 1 *hold except that the condition* (3) *is replaced by*

$$\begin{aligned} V(t + \lambda^{-1}, x + \lambda^{-1} f(t, x)) \leqq{} & V(t, x)(1 + \lambda^{-1} L(t)) \\ & + \lambda^{-1} W(t, V(t, x) e^{\alpha(t)}) e^{-\alpha(t)} + o(\lambda^{-1}), \end{aligned} \tag{5}$$

Received by the editors November 24, 1961 and, in revised form, March 9, 1962.

[1] This work was supported by the Office of Naval Research. The author's thanks are due to the referee for his helpful suggestions and to Professor E. A. Coddington for all the help received at the University of California at Los Angeles.

where $L(t)$ *is continuous for* $t \in I$ *and* $\alpha(t) = -\int_{t_0}^{t} L(s)ds$. *Then the inequality* (4) *is replaced by*

$$V(t, x(t))e^{\alpha(t)} \leqq r(t), \qquad (t \geqq t_0). \tag{6}$$

It is clear that Theorem 2 includes Theorem 1 and hence we prove Theorem 2.

PROOF OF THEOREM 2. Let $x(t)$ be any solution of (1) such that $V(t_0, x_0) \leqq r_0$. Define $m(t) = V(t, x(t))e^{\alpha(t)}$. Then $m(t_0) \leqq r_0$, since $\alpha(t_0) = 0$. As $V(t, x)$ is assumed to satisfy a Lipschitz condition, we have, for small $h > 0$,

$$m(t + h) - m(t) \leqq ke^{\alpha(t+h)} | \epsilon | h + e^{\alpha(t+h)}V(t + h, x(t) + hf(t, x(t))) - e^{\alpha(t)}V(t, x(t)),$$

where the vector ϵ tends to zero as h tends to zero and k is the Lipschitz constant. Now taking $h = \lambda^{-1}$ and using (5), one obtains

$$m(t + h) - m(t) \leqq ke^{\alpha(t+h)} | \epsilon | h + V(t, x(t))[e^{\alpha(t+h)} - e^{\alpha(t)}] + he^{\alpha(t+h)}[L(t)V(t, x(t)) + W(t, m(t))e^{-\alpha(t)} + o(h)],$$

which in its turn yields the inequality

$$\limsup_{h \to 0+} [m(t + h) - m(t)]/h \leqq W(t, m(t)).$$

This is sufficient to prove the result (6) as far as $x(t)$ exists, following the argument used in the lemma in [6].

Now suppose that $x(t)$ cannot be continued as far as $r(t)$ exists. Then there is a positive number t_1 such that $x(t)$ cannot be extended to the closed interval $t_0 \leqq t \leqq t_1$. This implies that there cannot exist an increasing sequence $\{t_n\}$ tending to t_1 such that $|x(t_n)|$ is bounded, which means that $|x(t_n)| \to \infty$ as $t \to t_1 - 0$. Since $V(t, x) \to \infty$ as $|x| \to \infty$, we get from (6) that $r(t_1 - 0) \to \infty$. This contradiction proves that $x(t)$ exists as far as $r(t)$ exists in view of a result of Wintner [10].

REMARK. We observe that $W(t, r)$ need not be non-negative. Taking $V(t, x) = |x|$ and $W(t, r) = k(t)r$ or $k(t)g(r)$, where $k(t)$ is continuous and $g(r) > 0$ for $r > 0$, the upper bounds referred to in [1; 2; 7; 8] can be obtained from Theorem 1 without demanding as much. In that case, condition (2) reduces to

$$| x + \lambda^{-1}f(t, x) | \leqq | x | + \lambda^{-1}k(t)g(| x |) + o(\lambda^{-1}),$$

which is weaker than the corresponding condition, viz;

$$| f(t, x) | \leqq k(t)g(| x |).$$

Obviously the latter condition demands $k(t)g(r)$ to be non-negative.

We also note that Theorems 1 and 2 contain the work of Conti [**3**].

The condition (2) is not strong enough to yield the lower bound referred to in [**7**; **8**]. We state the following result to that effect.

THEOREM 3. *Let the condition* (2) *of Theorem* 1 *be replaced by*

$$V(t + \lambda^{-1}, x + \lambda^{-1}f(t, x)) \geqq V(t, x) - \lambda^{-1}W(t, V(t, x)) + o(\lambda^{-1}).$$

Then, as long as $s(t) \geqq 0$ *and* $x(t)$ *exists*

$$V(t, x(t)) \geqq s(t),$$

where $s(t)$ *is the minimal solution of* $r' = -W(t, r)$, $s(t_0) \leqq V(t_0, x_0)$.

2. Consider the differential equation

$$x' = A(t)x + F(t, x) = f(t, x) \text{ say,} \tag{1'}$$

where $A(t)$ is a continuous $n \times n$ matrix. It is easy to show that by using our results one can generalize some known results pertaining to the above equation.

THEOREM 4. *Suppose that the assumptions of Theorem* 1 *are satisfied. Let*

$$b(|x|) \leqq V(t, x), \tag{7}$$

where $b(r)$ *is continuous, increasing in* r *and* $b(r) > 0$ *for* $r > 0$. *Then, if the scalar differential equation* (2) *is* (i) *stable*; (ii) *asymptotically stable, the system* (1′) *is* (i) *stable*; (ii) *asymptotically stable, respectively.*

PROOF. For any $\epsilon > 0$, if $|x| = \epsilon$, we have from (7), $b(\epsilon) \leqq V(t, x)$. If equation (2) is stable, given $b(\epsilon)$ and $t_0 \geqq 0$, there exists a $d = d(t_0, \epsilon)$ such that $r(t) < b(\epsilon)$ for $t \geqq t_0$, whenever $r_0 \leqq d$. Let $x(t)$ be any solution of (1′) satisfying $V(t_0, x_0) \leqq r_0 \leqq d$. Then we derive $|x_0| \leqq b^{-1}(d) = \gamma$ say. For such solutions, we have from Theorem 1

$$V(t, x(t)) \leqq r(t), \qquad (t \geqq t_0). \tag{8}$$

If possible, let $|x(t_1)| = \epsilon$ for some $t = t_1 > t_0$. Then one gets

$$b(\epsilon) \leqq V(t_1, x(t_1)) \leqq r(t_1) < b(\epsilon),$$

a contradiction which proves the stability of the system (1′).

To prove asymptotic stability, suppose if possible, $|x(t_n)| > \eta$, where $\{t_n\}$ is a divergent sequence and $\eta > 0$ is arbitrary. Then one obtains, as before,

$$b(\eta) \leqq V(t_n, x(t_n)) \leqq r(t_n).$$

Since equation (2) is asymptotically stable, $r(t_n) \to 0$ as $t_n \to \infty$. This implies a contradiction because $b(\eta) > 0$. Hence $x(t) \to 0$ as $t \to \infty$ and the proof is complete.

THEOREM 5. *Let the assumptions of Theorem* 2 *be satisfied. Let the conditions corresponding to* $b(r)$ *and* (7) *also hold. Suppose that* $\alpha(t) \to \infty$ *as* $t \to \infty$. *Then, if the equation* (2) *is stable, the system* (1′) *is asymptotically stable.*

PROOF. Following the same argument as in Theorem 4 we have now to replace (8) by

$$V(t, x(t))e^{\alpha(t)} \leqq r(t), \qquad (t \geqq t_0).$$

Let $\{t_n\}$ be a divergent sequence and if possible, $|x(t_n)| > \eta$, where $\eta > 0$ is arbitrary. Then one gets

$$e^{\alpha(t_n)} b(\eta) < b(\epsilon).$$

Since $\alpha(t_n) \to \infty$ as $t_n \to \infty$, this leads to a contradiction and proves the result.

The above theorems generalize some results of Halany [**5**] and Santoro [**3**].

REMARK. Since the previous considerations demand, as was pointed out, $W(t, r)$ to be non-negative, which implies that the solutions $r(t)$ of (2) are nondecreasing as t increases, one has limitations in assuming the properties that $r(t)$ should satisfy. For instance, $r(t) \to 0$ as $t \to \infty$ is impossible, as we have assumed in Theorem 4. The practical importance of our approach can be seen from the special case. Suppose that $V(t, x) = x \cdot x$. Then it is enough to take $L(t) = 2\lambda(t)$, where $\lambda(t)$ is the largest eigenvalue of $\frac{1}{2}[A(t) + A^*(t)]$, $A^*(t)$ being the transpose of $A(t)$, and $x \cdot F(t, x) \leqq k(t) x \cdot x$. Such a choice works in both the theorems above, since $\lambda(t)$ and $k(t)$ need not be positive for all $t \geqq t_0$.

Theorem 1 can be used in a slightly different way so as to yield another type of information regarding the solutions of (1′).

Let $U(t)$ be the matrix solution of $U'(t) = A(t)U(t)$, $U(t_0) =$ unit matrix. Setting $x = U(t)y$ and using the method of variation of constants, it is easy to obtain the differential equation

$$y' = U^{-1}(t) F(t, U(t)y) = f(t, y) \text{ say.}$$

If the assumptions of Theorem 1 are satisfied for this $f(t, y)$, we have immediately

$$V(t, y(t)) \leqq r(t), \qquad (t \geqq t_0).$$

If further $V(t, x) = |x|$, one obtains from the definition of y that

$$|x(t)| \leqq r(t) |U(t)|, \qquad (t \geqq t_0),$$

where $x(t)$ is any solution of (1′) with $|x_0| \leqq r_0$. This implies that the behaviour of solutions of the perturbed system depends on that of the unperturbed system, if $r(t)$ is bounded. Such a result was obtained by Golomb [4] under stronger assumptions.

References

1. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York, 1953.

2. I. Bihari, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Math. Acad. Sci. Hungar. **7** (1956), 81–94.

3. R. Conti, *Sulla prolungabilita delle soluzioni di un sistema di equazioni differenzial ordinarie*, Boll. Un. Mat. Ital. **11** (1956), 510–514.

4. M. Golomb, *Bounds for solutions of non-linear differential equations*, Arch. Rational Mech. Anal. **1** (1958), 272–282.

5. A. Halany, *Quelques observations sur la stabilité asymptotique*, An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 9, 31–38.

6. V. Lakshmikanth, *On the boundedness of solutions of non-linear differential equations*, Proc. Amer. Math. Soc. **8** (1957), 1044–1048.

7. ———, *Upper and lower bounds of the norm of solutions of differential equations*, Proc. Amer. Math. Soc. **13** (1962), 615–616.

8. C. E. Langenhop, *Bounds on the norm of a solution of general differential equation*, Proc. Amer. Math. Soc. **11** (1960), 795–99.

9. P. Santoro, *Sulla stabilita uniforme e asintotica uniforme in prima approssimazione*, Atti. Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 336–341.

10. A. Wintner, *The infinities in the non-local existence problem of ordinary differential equations*, Amer. J. Math. **68** (1946), 173–178.

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