A COMPACT TOPOLOGY FOR A LATTICE[1]

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Introduction. In this paper we shall study a compact intrinsic topology for a lattice and obtain a few relationships between this topology and certain well-known intrinsic topologies for lattices. We obtain as a result the fact that for a large class of lattices, compactness of the order topology implies that our compact topology and the order topology coincide.

Let L be a lattice and $\{x_a\}$, a net in L. We define the limit inferior, $L_*\{x_a\} = \bigvee_a \bigwedge_{b \geqq a} x_b$, and the limit superior, $L^*\{x_a\} = \bigwedge_a \bigvee_{b \geqq a} x_b$. Then, provided they exist, $L_*\{x_a\} \leqq L^*\{x_a\}$. If $L_*\{x_a\} = L^*\{x_a\} = x$, we say that the net $\{x_a\}$ order converges to x. Let C be a subset of L. C is said to be order closed iff no net in C order converges to a point outside of C. The collection of order closed sets comprises the closed sets for a topology for L. We call this topology the order topology for L and designate it by $O(L)$.

The collection of sets of the form $\{x : x \leqq c\}$ and $\{x : x \geqq c\}$ for $c \in L$ forms the sub-base of the closed sets of a weaker topology called the interval topology and designated by $I(L)$. It is known that for any lattice L, L is complete iff L with the interval topology is compact. (See G. Birkhoff's *Lattice theory*.)

The complete topology. DEFINITION 1. Let L be a lattice and $\mathcal{C}$, the collection of all complete subsets of L. Then $\mathcal{C}$ is a subbase of the closed sets for a topology which we shall denote by $K(L)$. The topology $K(L)$ will be called the *complete topology* for L.

LEMMA 2. *Let L be a lattice and $\mathfrak{N}$, a nest of nonempty complete subsets of L. Then $\bigcap \mathfrak{N} \neq \emptyset$.*

PROOF. Let S denote the set, $\{\bigvee N : N \in \mathfrak{N}\}$. For the following discussion we consider a fixed set N_0 in $\mathfrak{N}$. We divide the proof up into several remarks.

REMARK (a). Define $S_0 = \{s : s \in S \text{ and } s \leqq \bigvee N_0\}$. Then, $S_0 \subset N_0$.

PROOF. Consider any element $s \in S_0$. If $s = \bigvee N_0$ then $s \in N_0$, since N_0 is complete. Suppose $s < \bigvee N_0$. Then associated with s is a member N_s of $\mathfrak{N}$ such that $s = \bigvee N_s$. Then since $\mathfrak{N}$ is a nest, either $N_0 \subset N_s$ or $N_s \subset N_0$. If $N_0 \subset N_s$ then clearly, $\bigvee N_0 \leqq \bigvee N_s = s$, contrary to hypothesis. Hence $N_s \subset N_0$, and since N_0 is complete, $\bigvee N_s = s \in N_0$.

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REMARK (b). S is linearly ordered.

PROOF. Let s and t be elements of S. Associated with s and t are sets N_s and N_t in $\mathfrak{N}$ such that $s=\bigvee N_s$ and $t=\bigvee N_t$. Either $N_s\subset N_t$ or $N_t\subset N_s$. In the former case $s\leqq t$, in the latter, $t\leqq s$.

REMARK (c). Both S and S_0 possess infimums, and $\bigwedge S=\bigwedge S_0$.

PROOF. By Remark (a), S_0 is contained in a complete subset of L and hence, S_0 possesses an infimum, $\bigwedge S_0$. Consider any $x\in S$. By Remark (b), we may note that either $x\leqq \bigvee N_0$ or $x>\bigvee N_0$. If $x>\bigvee N_0$ then since $\bigvee N_0\in S_0$, we have that $\bigwedge S_0\leqq \bigvee N_0<x$. Clearly if $x\leqq \bigvee N_0$, $x\in S_0$ and hence, $\bigwedge S_0\leqq x$. We conclude that $\bigwedge S_0$ is a lower bound for S. Now suppose that y is any lower bound for S. Then since $S_0\subset S$, y is a lower bound for S_0 or, $y\leqq\bigwedge S_0$. Hence $\bigwedge S_0$ is the greatest lower bound for S or, $\bigwedge S_0=\bigwedge S$.

We now continue with our proof. N_0 is a complete subset of L. By Remark (a), $S_0\subset N_0$ and hence, $\bigwedge S_0\in N_0$. But by Remark (c), $\bigwedge S_0=\bigwedge S$ and hence, $\bigwedge S\in N_0$. N_0 was chosen as an arbitrary member of $\mathfrak{N}$ and it was found that $\bigwedge S\in N_0$. Hence $\bigwedge S\in\bigcap\mathfrak{N}$ or, $\bigcap\mathfrak{N}\neq\varnothing$.

THEOREM 3. *Let $\mathfrak{A}$ be a collection of complete subsets of a lattice L such that $\mathfrak{A}$ has the finite intersection property. Then,* $\bigcap\mathfrak{A}\neq\varnothing$.

PROOF. We use transfinite induction on the cardinality of $\mathfrak{A}$. Clearly the theorem holds if $\mathfrak{A}$ is finite. Suppose that the theorem holds if the cardinality of $\mathfrak{A}$ is less than some fixed cardinal number α. Let Γ be the set of all ordinal numbers less than the first ordinal number of cardinality α. Assume the cardinality of Γ is the same as that of $\mathfrak{A}$. Hence we can index $\mathfrak{A}$ with Γ. Then for each ordinal $n\in\Gamma$ we consider the set A_n in $\mathfrak{A}$ to which n corresponds. Define the set $C_n=\bigcap\{A_i: i\in\Gamma \text{ and } i\leqq n\}$. Then C_n is not empty because it is the intersection of a collection of complete sets with the finite intersection property, and this collection has cardinality less than α. Thus the collection $\{C_n: n\in\Gamma\}$ forms a nest of nonempty complete subsets of L. Hence by Lemma 2, $\bigcap\{C_n: n\in\Gamma\}=\bigcap\mathfrak{A}\neq\varnothing$.

THEOREM 4. *Let L be a lattice with the complete topology, $K(L)$. Then L is compact.*

PROOF. The lattice L with the topology $K(L)$ satisfies the following property: There exists a subbase of the closed sets for $K(L)$ such that every subcollection of this subbase with the finite intersection property has a nonempty intersection. But this condition is necessary and sufficient for L with the topology $K(L)$ to be compact. (For a proof see O. Frink's *Topology in lattices*.)

THEOREM 5. *Let L be a lattice. Then $K(L)\subset O(L)$, and $I(L)\subset K(L)$ iff L is complete.*

PROOF. Clearly every complete subset of L is order closed and hence, $K(L) \subset O(L)$. If $I(L) \subset K(L)$, then since $K(L)$ is compact, $I(L)$ is compact and hence, L is complete. If on the other hand, L is complete, each closed ray of the form $\{x : x \leqq c\}$ or $\{x : x \geqq c\}$ is complete and hence, $I(L) \subset K(L)$.

Let L be a complete lattice and A, a nonempty subset of L. Then by A^q we shall mean the smallest complete subset of L containing A. If $\{x_a\}$ is a net in L, by $\{x_a\}^q$ we shall mean the smallest complete subset of L containing the range of the net. The following is a characterization of topological convergence in a complete lattice with respect to the complete topology, $K(L)$.

THEOREM 6. *Let L be a complete lattice and $\{x_a\}$, a net in L. Then $\{x_a\}$ topologically converges to a point x in L with respect to the complete topology, $K(L)$, iff for each subnet $\{y_c\}$ of $\{x_a\}$, $x \in \{y_c\}^q$.*

PROOF. Suppose $\{x_a\}$ converges to a point x in L with respect to the complete topology. Let $\{y_c\}$ be an arbitrary subnet of $\{x_a\}$. Then $\{y_c\}$ converges to x. The set $\{y_c\}^q$ is closed and contains the net $\{y_c\}$. Hence, $x \in \{y_c\}^q$.

Now suppose that $\{x_a\}$ does not converge to x. Then there exists a complete subset C of L such that $\{x_a\}$ is frequently in C and $x \notin C$. Hence there exists a subnet $\{z_b\}$ of $\{x_a\}$ contained in C. Therefore $\{z_b\}^q$ is contained in C. Hence $x \notin \{z_b\}^q$.

COROLLARY 7. *Let $\{x_a\}$ be a net in a complete lattice L. Then the set of all elements in L to which $\{x_a\}$ converges with respect to $K(L)$ is a complete subset of L.*

COROLLARY 8. *Let L be a complete lattice and $\{x_a\}$, a net in L. Then if $\{x_a\}$ topologically converges to a point x in L with respect to the complete topology, $K(L)$, $L_*\{x_a\} \leqq x \leqq L^*\{x_a\}$.*

PROOF. We first note that for A, a nonempty subset of L, $\bigvee A = \bigvee A^q$, and $\bigwedge A = \bigwedge A^q$.

For any element a of the directed set of $\{x_a\}$, $x \in \{x_b : b \geqq a\}^q$. Hence, for any a, $\bigwedge_{b \geqq a} x_b \leqq x$. Therefore, $\bigvee_a \bigwedge_{b \geqq a} x_b = L_*\{x_a\} \leqq x$. Dually, $x \leqq L^*\{x_a\}$.

THEOREM 9. *Let L be a complete lattice and $T(L)$, any compact topology for L. Then if every complete subset of L is closed with respect to the topology $T(L)$, $T(L)$ is contained in the order topology, $O(L)$.*

PROOF. If every complete subset of L is closed with respect to the topology $T(L)$, we have that $K(L) \subset T(L)$. Hence, topological convergence with respect to $T(L)$ implies topological convergence with respect to $K(L)$. Therefore by the previous corollary, if $\{x_a\}$ is a net in L and $\{x_a\}$ converges to a point x with respect to the topology

$T(L)$, $L_*\{x_a\} \leqq x \leqq L^*\{x_a\}$.

Now let C be any subset of L such that C is not closed with respect to $O(L)$. Then there exists a net, $\{x_a\}$ in C which order converges to a point x, not in C. But since L is compact with respect to $T(L)$, there exists a subnet $\{y_b\}$ of $\{x_a\}$ and an element y in L such that $\{y_b\}$ topologically converges to y with respect to $T(L)$. Hence, $L_*\{x_a\} \leqq L_*\{y_b\} \leqq y \leqq L^*\{y_b\} \leqq L^*\{x_a\}$. But $L_*\{x_a\} = L^*\{x_a\} = x$. Hence, $x = y$. Therefore, $y \notin C$. Thus there exists a net in C which converges to a point, y, outside of C. Therefore C is not closed in the $T(L)$ topology. Therefore, $T(L) \subset O(L)$.

THEOREM 10. *Let L be any lattice which satisfies the following condition: If $\{x_a\}$ is a net in L which topologically converges to a point x with respect to the order topology, then there exists a subnet of $\{x_a\}$ which order converges to x.*

Then if L is compact with respect to its order topology, $O(L)$, it follows that $O(L) = K(L)$.

PROOF. Suppose that L satisfies the above condition and suppose further that L is compact with respect to $O(L)$. Then since $I(L) \subset O(L)$, $I(L)$ is compact and therefore, L is complete.

Let C be a subset of L such that C is not closed with respect to $K(L)$. Then there exists a net $\{x_a\}$ in C and a point $x \notin C$ such that $\{x_a\}$ converges to x with respect to the topology $K(L)$. Since L is compact with respect to $O(L)$, there exists a subnet $\{y_b\}$ of $\{x_a\}$ and a point y in L such that $\{y_b\}$ topologically converges to y with respect to the order topology. By hypothesis, there exists a subnet $\{z_c\}$ of $\{y_b\}$, and hence of $\{x_a\}$, which order converges to y. The net $\{z_c\}$ topologically converges to x with respect to $K(L)$. Hence by Corollary 8, $L_*\{z_c\} \leqq x \leqq L^*\{z_c\}$. However, $L_*\{z_c\} = L^*\{z_c\} = y$. Therefore, $x = y$. Hence, $y \notin C$. We conclude that C is not closed in the order topology for L. Hence, $O(L) \subset K(L)$. But $K(L) \subset O(L)$. Hence the theorem follows.

For any lattice L it can be shown that if L is Hausdorff with respect to the complete topology for L then L is a complete lattice. The following questions remain unanswered:

1. Does Theorem 10 hold for arbitrary lattices?

2. For any lattice L, is a necessary and sufficient condition for $O(L)$ to be compact that $K(L)$ be Hausdorff?

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