OPEN 3-MANIFOLDS WHICH ARE SIMPLY CONNECTED AT INFINITY

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A triangulated open manifold M will be called 1-*connected at infinity* if each compact subset C of M is contained in a compact polyhedron P in M such that $M-P$ is connected and simply connected. Stallings has shown that, if M is a contractible open combinatorial manifold which is 1-connected at infinity and is of dimension $n \geqq 5$, then M is piecewise-linearly homeomorphic to Euclidean n-space E^n [5].

THEOREM 1. *Let M be a contractible open 3-manifold, each of whose compact subsets can be imbedded in E^3. If M is 1-connected at infinity, then M is homeomorphic to E^3.*

Notice that, in order to prove the 3-dimensional Poincaré conjecture, it would suffice to prove Theorem 1 without the hypothesis that each compact subset of M can be imbedded in E^3. For, if $\overline{M}$ is a simply connected closed 3-manifold and p is a point of $\overline{M}$, then $\overline{M}-p$ is a contractible open 3-manifold which is clearly 1-connected at infinity. Conversely, if the 3-dimensional Poincaré conjecture were known, then the hypothesis that each compact subset of M can be imbedded in E^3 would be unnecessary.

All spaces and mappings in this paper are considered in the polyhedral or piecewise-linear sense, unless otherwise stated. As usual, by an open n-manifold is meant a noncompact connected space triangulated by a countable simplicial complex without boundary, such that the link of each vertex is piecewise-linearly homeomorphic to the usual $(n-1)$-sphere.

PROOF OF THEOREM 1. Let X be an arbitrary compact subset of M, and let Y be a connected compact subset of M which contains X. Using the fact that M is 1-connected at infinity, choose a compact polyhedron P such that $Y \subset P \subset M$ with $M-P$ connected and simply connected. If N is a regular neighborhood of P which contains P in its interior, then $M-N$ is also connected and simply connected, and the component N_0 of N which contains Y is a (connected) compact orientable 3-manifold with boundary [6]. Since M is contractible, and since the boundary S_0 of N_0 separates M into exactly two com-

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ponents Int N_0 and $M-N_0$, S_0 is connected by Alexander duality.

Since $M-N$ is simply connected, the 1-dimensional Betti number of N is zero (also by Alexander duality). But the 1-dimensional Betti number of a bounded orientable 3-manifold is at least as large as the sum of the genera of its boundary surfaces [**4**, p. 223]. It follows that S_0 is a 2-sphere. The assumption that each compact subset of E^3 can be imbedded in E^3 now implies that N_0 is a 3-cell which contains X in its interior.

Since each compact subset of M lies interior to a 3-cell in M it follows easily that M is the union of a sequence $\{C_n\}_1^\infty$ of 3-cells, with $C_n \subset \operatorname{Int} C_{n+1}$, $n=1, 2, \cdots$. A theorem of Brown now applies to show that M is homeomorphic to E^3 [**1**].

Theorem 1 will next be generalized by suppressing the restriction that M be contractible and relaxing the restriction that it be 1-connected at infinity. Let an open manifold U be called *simply connected at infinity* if each compact subset B of U is contained in a compact polyhedron Q in U such that each component of $U-Q$ is simply connected. By a *punctured cube* will be meant a space obtained from a 3-sphere by deleting the interiors of a finite (positive) number of mutually disjoint polyhedral 3-cells.

LEMMA 1. *Let U be an open 3-manifold which is simply connected at infinity, and such that each compact subset of U can be imbedded in E^3. Then each compact subset of U lies interior to a punctured cube in U.*

PROOF. Let A be an arbitrary compact subset of U, and let B be a connected compact subset of U containing A. Since U is simply connected at infinity, there is a compact polyhedron Q in U containing B, such that each component of $U-Q$ is simply connected. If N is a regular neighborhood of Q containing Q in its interior, then each component of $U-N$ is also simply connected, and the component N_0 of N which contains B is a (connected) compact orientable 3-manifold with boundary [**6**].

It will be shown that the fact that each component of $U-N$ is simply connected implies that N_0 lies in a punctured cube in U. The proof of this is by induction on the sum g of the genera of the boundary surfaces of N_0. If $g=0$, then the hypothesis that each compact subset of U can be imbedded in E^3 implies that N_0 itself is a punctured cube. Now assume that the conclusion follows if $g<k$, where $k \geqq 1$, and let the sum of the genera of the boundary surfaces of N_0 be k.

If S is a closed orientable 2-manifold of positive genus on the boundary of N_0, let J be a simple closed curve encircling one of the handles of S. Then the Dehn lemma [**2**] gives a 2-cell D with Bd $D=J$

and $\operatorname{Int} D \subset U - N$, since each component of $U - \operatorname{Int} N$ is simply connected.

It is first shown that each component of $V = (U - N) - D$ is simply connected. Let K be a simple closed curve in V and let f be a piecewise-linear map of a 2-cell E into $U - N$ such that $f | \operatorname{Bd} E$ is a homeomorphism of $\operatorname{Bd} E$ onto K and such that f is in general position with respect to D, in the sense that each component of $f^{-1}(D)$ is a simple closed curve. Let K^1 be an "inner" one of these simple closed curves, bounding the subdisk E^1 of E. Then K^1 can be eliminated by first redefining f on E^1 and then deforming the new image of E^1 slightly away from D. After a finite number of steps of this kind, it is seen that K can be shrunk to a point in V.

Now thicken the 2-cell D to form a 3-cell C such that $S \cap \operatorname{Bd} C$ is an annular ring R with $C - R \subset U - N$. Then each component of $U - (N \cup C)$ is simply connected, and the sum of the genera of the boundary surfaces of $N_0 \cup C$ is $k - 1$. By induction U therefore contains a punctured cube containing $N_0 \cup C$ and hence containing A in its interior.

The following elementary lemma is easily proved.

LEMMA 2. *Let A and B be punctured cubes with $A \subset \operatorname{Int} B$ and let C and D be 3-cells with $C \subset \operatorname{Int} D$. Suppose that S and T are components of* $\operatorname{Bd} A$ *and* $\operatorname{Bd} B$, *respectively, such that S separates* $\operatorname{Int} A$ *and T in B. If f is a piecewise-linear homeomorphism of A into C such that $f(S) = \operatorname{Bd} C$, then there is a piecewise-linear homeomorphism g of B into D such that $g(T) = \operatorname{Bd} D$ and $g/A = f$.*

THEOREM 2. *Let the open 3-manifold U be the union of a sequence $\{A_i\}_1^\infty$ of punctured cubes, with $A_i \subset \operatorname{Int} A_{i+1}$, $i = 1, 2, \cdots$. Then there is a totally disconnected subset Y of E^3 such that U and $E^3 - Y$ are homeomorphic.*

PROOF. The collection $\{A_i\}_1^\infty$ is first subjected to a sequence of alterations as follows. In the first step, a new punctured cube A_1^1 interior to A_2 is obtained from A_1 by adding to A_1 each component of $A_2 - A_1$ which contains no component of $\operatorname{Bd} A_2$ (the closure of each such component of $A_2 - A_1$ is a 3-cell). Each component of $\operatorname{Bd} A_1^1$ will then separate $\operatorname{Int} A_1^1$ and some component of $\operatorname{Bd} A_2$ in A_2.

In the second step, the punctured cube A_2^2 interior to A_3 is obtained from A_2 by adding to A_2 each component of $A_3 - A_2$ which contains no component of $\operatorname{Bd} A_3$, and then A_1^2 is obtained from A_1^1 by adding to A_1^1 each component of $A_2^2 - A_1^1$ which contains no component of $\operatorname{Bd} A_2^2$. Now each component of $\operatorname{Bd} A_1^2$ separates $\operatorname{Int} A_1^2$ from some component of Bd A_2^2 in A_2^2, and each component of Bd A_2^2 separates Int A_2^2 and some component of Bd A_3 in A_3.

Suppose that the punctured cubes $A_1^{i-1}, A_2^{i-1}, \cdots, A_{i-1}^{i-1}$ are the result of the first $i-1$ steps in this process. In the ith step the punctured cube A_i^i is obtained from A_i by adding to A_i each component of $A_{i+1}-A_i$ which contains no component of Bd A_{i+1}; then A_{i-1}^i is obtained from A_{i-1}^{i-1} by adding to A_{i-1}^{i-1} each component of $A_i^i - A_{i-1}^i$ which contains no component of Bd A_i^i, and so on; finally the punctured cube A_1^i is obtained from A_1^{i-1} by adding to A_1^{i-1} each component of $A_2^i - A_1^{i-1}$ which contains no component of Bd A_2^i. Now the punctured cubes $A_1^i, \cdots, A_i^i$ satisfy the condition that each component of Bd A_j^i, $j<i$, separates Int A_j^i and some component of Bd A_{j+1}^i in A_{j+1}^i, and each component of Bd A_i^i separates Int A_i^i and some component of Bd A_{i+1} in A_{i+1}.

This process is continued by induction. Notice that $A_i^m = A_i^n$ if m and n are sufficiently large, $i=1, 2, \cdots$. Consequently the result of this sequence of alterations is a new sequence $\{B_i\}_1^\infty$ of punctured cubes such that (1) $U=\bigcup_{i=1}^\infty B_i$, (2) $B_i \subset \operatorname{Int} B_{i+1}$ for each i, and (3) each component of Bd B_i separates Int B_i and some component of Bd B_{i+1} in B_{i+1}, $i=1, 2, \cdots$.

Now let E^3 be expressed as the union of a sequence $\{C_i\}_1^\infty$ of polyhedral 3-cells such that $C_i \subset \operatorname{Int} C_{i+1}$, $i=1, 2, \cdots$. Let S_1 be any component of Bd B_1 and, S_{i-1} having been defined as a component of Bd B_{i-1}, let S_i be a component of Bd B_i such that S_{i-1} separates Int B_{i-1} and S_i in B_i.

Then use Lemma 2 to define by induction a sequence $\{g_i\}_1^\infty$ of maps such that, for each i, (1) g_i is a piecewise-linear homeomorphism of B_i into C_i, (2) $g_i(S_i) = \operatorname{Bd} C_i$, and (3) $g_i | B_{i-1} = g_{i-1}$. Finally define a homeomorphism f of U into E^3 by setting $f(x) = g_i(x)$ if $x \in B_i$.

If F_i is the closure of $C_i - g_i(B_i)$, then clearly F_i is the union of a finite number of mutually disjoint 3-cells. If $X = \bigcap_{i=1}^\infty F_i$, then $f(U) = E^3 - X$. Now let G be the decomposition space obtained from E^3 by shrinking each component of X to a point, and let h be the natural map of E^3 onto G, which is a homeomorphism on $E^3 - X$. Then $Y = h(X)$ is a totally disconnected subset of G, and it follows from [3] that G is homeomorphic to E^3. But hf is a homeomorphism of U onto $G - Y$, so the proof of Theorem 2 is complete.

THEOREM 3. *Let U be an open 3-manifold, each of whose compact subsets can be imbedded in E^3. If U is simply connected at infinity, then there is a totally disconnected subset Y of E^3 such that U and $E^3 - Y$ are homeomorphic.*

PROOF. Lemma 1 implies that U can be expressed as the union of an increasing sequence of punctured cubes, as in the hypotheses of Theorem 2.

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